The Busy Beaver Competition: a historical survey

Pascal MICHEL*

Équipe de Logique Mathématique, Institut de Mathématiques de Jussieu–Paris Rive Gauche,
UMR 7586, Bâtiment Sophie Germain, case 7012, 75205 Paris Cedex 13, France
and Université de Cergy-Pontoise, ESPE, F-95000 Cergy-Pontoise, France
michel@math.univ-paris-diderot.fr

Version 6
September 4, 2019

Abstract

Tibor Rado defined the Busy Beaver Competition in 1962. He used Turing machines to give explicit definitions for some functions that are not computable and grow faster than any computable function. He put forward the problem of computing the values of these functions on numbers 1, 2, 3, ... More and more powerful computers have made possible the computation of lower bounds for these values. In 1988, Brady extended the definitions to functions on two variables.

We give a historical survey of these works. The successive record holders in the Busy Beaver Competition are displayed, with their discoverers, the date they were found, and, for some of them, an analysis of their behavior.

We also survey the relations between busy beaver functions, the variants of their definitions, and the links with logical unprovability.

Keywords: Turing machine, busy beaver.

Mathematics Subject Classification (2010): *Primary* 03D10, *Secondary* 68Q05.

1 Introduction

1.1 Noncomputable functions

In 1936, Turing succeeded in making formal the intuitive notion of a function computable by a finite, mechanical, procedure. He defined what is now called a *Turing machine* and stated that a function on integers is intuitively computable if and only if it is computable by a Turing machine. Other authors, such as Church, Kleene, Post, and, later, Markov, defined other models of computation that turn out to compute the same functions as Turing machines do. See Soare (1996, 2007, 2009) for more details about the history of the *Church-Turing Thesis*,

*Corresponding address: 59 rue du Cardinal Lemoine, 75005 Paris, France.

as is now named the capture of the intuitive notion of computability by the formal notion of Turing machine.

Given a model of computation, a *noncomputable* function can easily be defined by *diagonalization*. The list of all computable functions is written, and then a function is defined such that it is distinct from each function in the list. Then this function is noncomputable. Such a definition by diagonalization leaves too much room in the choice of the list and in the choice of the values of the final function. What is needed is a function whose definition is simple, natural and without ambiguousness.

In 1962, Rado succeeded in providing a natural definition for noncomputable functions on the integers. He defined a *Busy Beaver* game, leading to two functions Σ and S which are still the best examples of noncomputable functions that one can give nowadays. The values $\Sigma(n)$ and $S(n)$ are defined by considering the finite set of carefully defined Turing machines with two symbols and n states, and picking among these machines those with some maximal behavior. It makes sense to compute the values $\Sigma(n)$, $S(n)$ of these functions on small integers $n = 1, 2, \ldots$. We have $\Sigma(1) = S(1) = 1$, trivially. Lin and Rado (1965) gave proofs for the values $\Sigma(2)$, $S(2)$, $\Sigma(3)$ and $S(3)$, and Brady (1983) did for $\Sigma(4)$ and $S(4)$. Only lower bounds had been provided for $\Sigma(5)$ and $S(5)$, by the works of Green, Lynn, Schult, Uhing and eventually Marxen and Buntrock. The lower bounds for $\Sigma(6)$ and $S(6)$ are still an ongoing quest.

The initial Busy Beaver game, as defined by Rado, used Turing machines with two symbols. Brady (1988) generalized the problem to Turing machines with k symbols, $k \geq 3$. He defined a function $S(n, k)$ such that $S(n, 2)$ is Rado's $S(n)$, and gave some lower bounds. Michel (2004) resumed the computation of lower bounds for $S(n, k)$ and another function $\Sigma(n, k)$, and the search is going on, with the works of Brady, Souris, Lafitte and Papazian, T. and S. Ligocki.

Since 2004, results are sent by email to Marxen and to Michel, who record them on their websites. This paper aims to give a published version of these records.

1.2 Big numbers

Consider Rado's functions S and Σ. Not only they are not computable, but they grow faster than any computable function. That is, for any computable function f, there exists an integer N such that, for all $n > N$, we have $S(n) > f(n)$. This property can be used to write big numbers. For example, if $S^k(n)$ denotes $S(S(\ldots S(n)\ldots))$, iterated k times, then $S^{9^9}(9)$ is a very big number, bigger than any number that was written with six symbols before the definition of the S function.

Bigger numbers can be obtained by defining functions growing much faster than Rado's busy beaver functions. A natural idea to get such functions is to define Turing machines of order k as follows. *Turing machines of order 1* are usual Turing machines without oracle, and, for $k \geq 2$, *Turing machines of order* k are Turing machines with oracle, where the oracle is the halting problem for Turing machines of order $k-1$. Then the *k-th busy beaver function* $B_k(n)$ is the maximum number of steps taken by a Turing machine of order k with n states and two symbols that stops when it is launched on a blank tape. So $B_1(n) = S(n)$, and $B_k(n)$ grows faster than any function computable by a Turing machine of order k.

Unfortunately, there is no canonical way to define a Turing machine with oracle, so Scott Aaronson, in his paper *Who can name the bigger number?* (see the website), asked for naturally defined functions growing as fast as the k-th busy beaver functions for $k \geq 2$. Such functions were found by Nabutovsky and Weinberger (2007). By using homology of groups, they defined a function growing as fast as the third busy beaver function, and another one growing as fast as the fifth busy beaver function. Michel (2010) went on studying these functions.

1.3 Contents

The paper is structured as follows.

1. Introduction.
2. Preliminaries.
3. Historical overview.
4. Historical survey (lower bounds for $S(n,k)$ and $\Sigma(n,k)$, and tables of the Turing machines that achieve these lower bounds).
5. Behaviors of busy beavers. We also display the relations between these behaviors and open problems in mathematics called Collatz-like problems and we resume some machines with non-Collatz-like behaviors. We also present pairs of machines that have the same behaviors, but not the same numbers of states and symbols.
6. Properties of the busy beaver functions and relations between $S(n)$ and $\Sigma(n)$.
7. Variants of busy beavers:
   - Busy beavers defined by 4-tuples.
   - Busy beavers whose head can stand still.
   - Busy beavers on a one-way infinite tape.
   - Two-dimensional busy beavers.
8. The methods.
9. Busy beavers and unprovability.

2 Preliminaries

There are many possible definitions for a Turing machine. We will follow the conventions chosen by Rado (1962) in his definition of functions Σ and S. A Turing machine has a tape, made of cells, infinite to the left and to the right. On each cell a *symbol* is written. There is a finite set $S = \{0, 1, \ldots\}$ of symbols. The symbol 0 is the *blank symbol*. A Turing machine has a tape head, which reads and writes symbols on the tape, and can move in both *directions* left or right, denoted by L and R. A Turing machine has a finite set of *states* $Q = \{A, B, \ldots\}$, plus a special state H, the *halting state*. A Turing machine has a *next move function*

$$\delta : Q \times S \longrightarrow (S \times \{L, R\} \times Q) \cup \{(1, R, H)\}.$$

If we have $\delta(q, a) = (b, d, p)$, then it means that, when the Turing machine is in state q and reads symbol a on the tape, then it writes symbol b instead of a on the cell currently read, it moves one cell in the direction $d \in \{L, R\}$, and it changes the state from q to p. Each application of next move function δ is a *step* of the computation. If $\delta(q, a) = (1, R, H)$, then, when the machine is in state q reading symbol a, it writes a 1, moves right, enters state H, and stops. We follow Rado (1962) in not allowing the center direction, that is in compelling the tape head to move left or right at each step. Like Rado, we keep the halting state H out of the set of states. We differ from Rado in not allowing transitions $\delta(q, a) = (b, d, H)$ with $b \neq 1$, $d \neq R$.

Note that such a machine is a universal model of computation. That is, any computable function on integers can be computed by a Turing machine as defined above. Initially, a finite string of symbols is written on the tape. It is called the input, and can be a code for an integer. All other cells contain the blank symbol. The tape head reads the leftmost symbol of the input and the state is the initial state A. Then the computation is launched according to the next move function. If it stops, by entering the halting state H, then the string of symbols written on the tape is the output, which can be a code for an integer. So a Turing machine defines a partial function on integers. Reciprocally, any computable partial function on integers can be computed by a Turing machine as defined above.

In order to define functions Σ and S, Rado (1962) considers Turing machines with n states and two symbols 0 and 1. His definitions can be easily extended to Turing machines with n states and k symbols, $k \geq 3$, as Brady (1988) does. We consider the set $TM(n, k)$ of Turing machines with n states and k symbols. With our definitions, it is a finite set with $(2kn + 1)^{kn}$ members. We launch each of these $(2kn + 1)^{kn}$ Turing machines on a blank tape, that is a tape with the blank symbol 0 in each cell. Some of these machines never stop. The other ones, that eventually stop, are called *busy beavers*, and they are competing in two competitions, for the maximum number of steps and for the maximum number of non-blank symbols left on the tape. Let $s(M)$ be the number of computation steps taken by the busy beaver M to stop. Let $\sigma(M)$ be the number of non-blank symbols left on the tape by the busy beaver M when it stops. Then the busy beaver functions are

$$S(n, k) = \max\{s(M) : M \text{ is a busy beaver with } n \text{ states and } k \text{ symbols}\},$$

$$\Sigma(n, k) = \max\{\sigma(M) : M \text{ is a busy beaver with } n \text{ states and } k \text{ symbols}\}.$$

For $k = 2$, we find Rado's functions $S(n) = S(n, 2)$ and $\Sigma(n) = \Sigma(n, 2)$.

Note that a permutation of the states, symbols or directions does not change the behavior of a Turing machine. The choice between machines that differ only by such permutations is settled by the following normalizing rule: when a Turing machine is launched on a blank tape, it enters states in the order A, B, $C, \ldots$, it writes symbols in the order 1, 2..., and it first moves right. So, normally, the first transition is $\delta(A, 0) = (1, R, B)$ or $\delta(A, 0) = (0, R, B)$.

Note about terminology and notations

Many names are used by authors: busy beaver game, busy beaver contest, busy beaver problem, busy beaver competition. All of them were first used early: game by Rado (1962), contest and problem by Rado (1963), competition by Green (1964).

What is exactly a busy beaver is rarely specified. Let us give some exceptions. For some authors, such as Green (1964) and Oberschelp et al. (1988), a busy beaver is any

Turing machine that participates to the busy beaver competition and halts. For others, such as Dewdney (1984) and Ben-Amram and Petersen (2002), a busy beaver is a winner of this competition. Rado (1962) called the winner a champion, and this term has been used sometimes afterwards.

The number of ones left on the tape by the Turing machine M when it stops is often called the *score* and denoted by $\sigma(M)$, since Rado (1962). It is called the *productivity* by Boolos and Jeffrey (1974), a term used again by Hertel (2009) and Harland (2013,2016). Harland uses the term *activity* for the number of moves of a Turing machine.

Almost all authors use the notations $\Sigma(n)$ and $S(n)$ for the busy beaver functions. Notable exceptions are: ones (n) and time(n) by Ben-Amram et al. (1996) and Ben-Amram and Petersen (2002); $bb(n)$ and $ff(n)$ by Harland (2013,2016).

3 Historical overview

The search for champions in the busy beaver competition can be roughly divided into the following stages. Note that, from the beginnings, computers have been tools to find good competitors, so better results follow more powerful computers.

First stage: Following the definitions. The definitions of the busy beaver functions $\Sigma(n)$ and $S(n)$ by Rado (1962) were quickly followed by conjectures and proofs for $n = 2, 3$, by Rado and Lin. Brady (1964) gave a conjecture for $n = 4$, and Green (1964) gave lower bounds for many values of n. Lynn (1972) improved these lower bounds for $n = 5, 6$. Brady proved his conjecture for $n = 4$ in 1974, and published the result in 1983. Details on this first stage can be found in the articles of Lynn (1972) and Brady (1983, 1988).

Second stage: Following the Dortmund contest. More results for $n = 5, 6$ followed the contest that was organized at Dortmund in 1983, and was wun by Schult. Uhing improved twice the result in 1984 and in 1986. Marxen and Buntrock began a search for competitors for $n = 5, 6$ in 1989. They quickly found a conjectural winner for $n = 5$, and went on finding many good machines for $n = 6$, up to 2001. Michel (1993) studied the behaviors of many competitors for $n = 5$, proving that they depend on well known open problems in number theory. Details on this second stage can be found in the articles of Dewdney (1984ab,1985ab), Brady (1988), and Marxen and Buntrock (1990). From 1997, results began to be put on the web, either on Google groups, or on personal websites.

Third stage: Machines with more than two symbols. As soon as 1988, Brady extended the busy beaver competition to machines with more than two symbols and gave some lower bounds. Michel (2004) resumed the search, and his lower bounds were quickly overtaken by those from Brady. Between 2005 and 2008, more than forty new machines, each one breaking a record, were found by two teams: the French one made of Grégory Lafitte and Christophe Papazian, and the father-and-son collaboration of Terry and Shawn Ligocki. Four new machines for the classical busy beaver competition of machines with 6 states and 2 symbols were also found, by the Ligockis and by Pavel Kropitz.

With the coming of the web age, researchers have faced two problems: how to announce results, and how to store them. In 1997, Heiner Marxen chose to post them on Google groups,

1963	Rado, Lin	$S(2,2) = 6$, $\Sigma(2,2) = 4$ $S(3,2) = 21$, $\Sigma(3,2) = 6$
1964	Brady	(4,2)-TM: $s = 107$, $\sigma = 13$
1964	Green	(5,2)-TM: $\sigma = 17$ (6,2)-TM: $\sigma = 35$ (7,2)-TM: $\sigma = 22{,}961$
1972	Lynn	(5,2)-TM: $s = 435$, $\sigma = 22$ (6,2)-TM: $s = 522$, $\sigma = 42$
1973	Weimann	(5,2)-TM: $s = 556$, $\sigma = 40$
1974	Lynn	(5,2)-TM: $s = 7{,}707$, $\sigma = 112$
1974	Brady	$S(4,2) = 107$, $\Sigma(4,2) = 13$
1983	Brady	(6,2)-TM: $s = 13{,}488$, $\sigma = 117$
1982	Schult	(5,2)-TM: $s = 134{,}467$, $\sigma = 501$ (6,2)-TM: $s = 4{,}208{,}824$, $\sigma = 2{,}075$
December 1984	Uhing	(5,2)-TM: $s = 2{,}133{,}492$, $\sigma = 1{,}915$
February 1986	Uhing	(5,2)-TM: $s = 2{,}358{,}064$
1988	Brady	(2,3)-TM: $s = 38$, $\sigma = 9$ (2,4)-TM: $s = 7{,}195$, $\sigma = 90$
February 1990	Marxen, Buntrock	**(5,2)-TM**: $s = 47{,}176{,}870$, $\sigma = 4{,}098$ (6,2)-TM: $s = 13{,}122{,}572{,}797$, $\sigma = 136{,}612$
September 1997	Marxen, Buntrock	(6,2)-TM: $s = 8{,}690{,}333{,}381{,}690{,}951$, $\sigma = 95{,}524{,}079$
August 2000	Marxen, Buntrock	(6,2)-TM: $s > 5.3 \times 10^{42}$, $\sigma > 2.5 \times 10^{21}$
October 2000	Marxen, Buntrock	(6,2)-TM: $s > 6.1 \times 10^{925}$, $\sigma > 6.4 \times 10^{462}$
March 2001	Marxen, Buntrock	(6,2)-TM: $s > 3.0 \times 10^{1730}$, $\sigma > 1.2 \times 10^{865}$

Table 1: Busy Beaver Competition from 1963 to 2001. In the last column, an (n,k)-Turing machine is a Turing machine with n states and k symbols. Number s is the number of steps, and number σ is the number of non-blank symbols left by the Turing machine when it stops. When (n,k)-TM is in bold type, the Turing machine is the current record holder. When values of $S(n,k)$ and $\Sigma(n,k)$ are indicated, the line refers to the proof that the functions have these values.

October 2004	Michel	(3,3)-TM: s = 40,737, σ = 208
November 2004	Brady	(3,3)-TM: s = 29,403,894, σ = 5,600
December 2004	Brady	(3,3)-TM: s = 92,649,163, σ = 13,949
February 2005	T. and S. Ligocki	**(2,4)-TM**: s = 3,932,964, σ = 2,050 (2,5)-TM: s = 16,268,767, σ = 4,099 (2,6)-TM: s = 98,364,599, σ = 10,574
April 2005	T. and S. Ligocki	(4,3)-TM: s = 250,096,776, σ = 15,008 (3,4)-TM: s = 262,759,288, σ = 17,323 (2,5)-TM: s = 148,304,214, σ = 11,120 (2,6)-TM: s = 493,600,387, σ = 15,828
July 2005	Souris	(3,3)-TM: s = 544,884,219, σ = 36,089
August 2005	Lafitte, Papazian	(3,3)-TM: s = 4,939,345,068, σ = 107,900 (2,5)-TM: s = 8,619,024,596, σ = 90,604
September 2005	Lafitte, Papazian	(3,3)-TM: s = 987,522,842,126, σ = 1,525,688 (2,5)-TM: σ = 97,104
October 2005	Lafitte, Papazian	(2,5)-TM: s = 233,431,192,481, σ = 458,357 (2,5)-TM: s = 912,594,733,606, σ = 1,957,771
December 2005	Lafitte, Papazian	(2,5)-TM: s = 924,180,005,181
April 2006	Lafitte, Papazian	(3,3)-TM: s = 4,144,465,135,614, σ = 2,950,149
May 2006	Lafitte, Papazian	(2,5)-TM: s = 3,793,261,759,791, σ = 2,576,467
June 2006	Lafitte, Papazian	(2,5)-TM: s = 14,103,258,269,249, σ = 4,848,239
July 2006	Lafitte, Papazian	(2,5)-TM: s = 26,375,397,569,930
August 2006	T. and S. Ligocki	(3,3)-TM: s = 4,345,166,620,336,565, σ = 95,524,079 (2,5)-TM: $s > 7.0 \times 10^{21}$, σ = 172,312,766,455

Table 2: Busy Beaver Competition from 2004 to 2006

June 2007	Lafitte, Papazian	$S(2,3) = 38$, $\Sigma(2,3) = 9$
September 2007	T. and S. Ligocki	(3,4)-TM: $s > 5.7 \times 10^{52}$, $\sigma > 2.4 \times 10^{26}$ (2,6)-TM: $s > 2.3 \times 10^{54}$, $\sigma > 1.9 \times 10^{27}$
October 2007	T. and S. Ligocki	(4,3)-TM: $s > 1.5 \times 10^{1426}$, $\sigma > 1.1 \times 10^{713}$ (3,4)-TM: $s > 4.3 \times 10^{281}$, $\sigma > 6.0 \times 10^{140}$ (3,4)-TM: $s > 7.6 \times 10^{868}$, $\sigma > 4.6 \times 10^{434}$ (3,4)-TM: $s > 3.1 \times 10^{1256}$, $\sigma > 2.1 \times 10^{628}$ (2,5)-TM: $s > 5.2 \times 10^{61}$, $\sigma > 9.3 \times 10^{30}$ (2,5)-TM: $s > 1.6 \times 10^{211}$, $\sigma > 5.2 \times 10^{105}$
November 2007	T. and S. Ligocki	(6,2)-TM: $s > 8.9 \times 10^{1762}$, $\sigma > 2.5 \times 10^{881}$ **(3,3)-TM**: $s = 119{,}112{,}334{,}170{,}342{,}540$, $\sigma = 374{,}676{,}383$ (4,3)-TM: $s > 7.7 \times 10^{1618}$, $\sigma > 1.6 \times 10^{809}$ (4,3)-TM: $s > 3.7 \times 10^{1973}$, $\sigma > 8.0 \times 10^{986}$ (4,3)-TM: $s > 3.9 \times 10^{7721}$, $\sigma > 4.0 \times 10^{3860}$ (4,3)-TM: $s > 3.9 \times 10^{9122}$, $\sigma > 2.5 \times 10^{4561}$ (3,4)-TM: $s > 8.4 \times 10^{2601}$, $\sigma > 1.7 \times 10^{1301}$ (3,4)-TM: $s > 3.4 \times 10^{4710}$, $\sigma > 1.4 \times 10^{2355}$ (3,4)-TM: $s > 5.9 \times 10^{4744}$, $\sigma > 2.2 \times 10^{2372}$ **(2,5)-TM**: $s > 1.9 \times 10^{704}$, $\sigma > 1.7 \times 10^{352}$ (2,6)-TM: $s > 4.9 \times 10^{1643}$, $\sigma > 8.6 \times 10^{821}$ (2,6)-TM: $s > 2.5 \times 10^{9863}$, $\sigma > 6.9 \times 10^{4931}$
December 2007	T. and S. Ligocki	(6,2)-TM: $s > 2.5 \times 10^{2879}$, $\sigma > 4.6 \times 10^{1439}$ (4,3)-TM: $s > 7.9 \times 10^{9863}$, $\sigma > 8.9 \times 10^{4931}$ (4,3)-TM: $s > 5.3 \times 10^{12068}$, $\sigma > 4.2 \times 10^{6034}$ **(3,4)-TM**: $s > 5.2 \times 10^{13036}$, $\sigma > 3.7 \times 10^{6518}$
January 2008	T. and S. Ligocki	**(4,3)-TM**: $s > 1.0 \times 10^{14072}$, $\sigma > 1.3 \times 10^{7036}$ **(2,6)-TM**: $s > 2.4 \times 10^{9866}$, $\sigma > 1.9 \times 10^{4933}$
May 2010	Kropitz	(6,2)-TM: $s > 3.8 \times 10^{21132}$, $\sigma > 3.1 \times 10^{10566}$
June 2010	Kropitz	**(6,2)-TM**: $s > 7.4 \times 10^{36534}$, $\sigma > 3.4 \times 10^{18267}$
March 2014	"Wythagoras"	**(7,2)-TM**: $s > \sigma > 10^{10^{10^{10^{18,705,352}}}}$

Table 3: Busy Beaver Competition since 2007

but it seems that the oldest reports are no longer available. From 2004, most results have been announced by sending them by email to several people (for example, the new machines with 6 states and 2 symbols found by Terry and Shawn Ligocki in November and December 2007 were sent by email to six persons: Allen H. Brady, Grégory Lafitte, Heiner Marxen, Pascal Michel, Christophe Papazian and Myron P. Souris). Storing results have been made on web pages (see websites list after the references). Brady has stored results on machines with 3 states and 3 symbols on his own website. Both Marxen and Michel have kept account of all results on their websites. Moreover, Marxen has held simulations, with four variants, of each discovered machine. Michel has held theoretical analyses of many machines.

4 Historical survey

4.1 Turing machines with 2 states and 2 symbols

- Rado (1963) claimed that $\Sigma(2,2)=4$, but that $S(2,2)$ was yet unknown.
- The value $S(2,2)=6$ was probably set by Lin in 1963. See

  `http://www.drb.insel.de/~heiner/BB/simTM22_bb.html`

  for a study of the winner by H. Marxen.

1963	Rado, Lin	$S(2,2)=6$	$\Sigma(2,2)=4$

The winner and some other good machines:

A0	A1	B0	B1	$s(M)$	$\sigma(M)$
1RB	1LB	1LA	1RH	6	4
1RB	1RH	1LB	1LA	6	3
1RB	0LB	1LA	1RH	6	3

4.2 Turing machines with 3 states and 2 symbols

- Soon after the definition of the functions S and Σ, by Rado (1962), it was conjectured that $S(3,2)=21$, and $\Sigma(3,2)=6$.
- Lin (1963) proved this conjecture and this proof was eventually published by Lin and Rado (1965). See studies by Heiner Marxen of the winners for the S function in

  `http://www.drb.insel.de/~heiner/BB/simTM32_bbS.html`

  and for the Σ function in

  `http://www.drb.insel.de/~heiner/BB/simTM32_bbO.html`

1963	Rado, Lin	$S(3,2)=21$	$\Sigma(3,2)=6$

The winners and some other good machines:

A0	A1	B0	B1	C0	C1	$s(M)$	$\sigma(M)$
1RB	1RH	1LB	0RC	1LC	1LA	21	5
1RB	1RH	0LC	0RC	1LC	1LA	20	5
1RB	1LA	0RC	1RH	1LC	0LA	20	5
0RB	1RH	0LC	1RA	1RB	1LC	17	4
0RB	1LC	1LA	1RB	1LB	1RH	16	5
1RB	1RH	0RC	1RB	1LC	1LA	14	6
1RB	1RC	1LC	1RH	1RA	0LB	13	6
1RB	1LC	1LA	1RB	1LB	1RH	13	6
0RB	1LC	1RC	1RB	1LA	1RH	13	5
1RB	1RA	1LC	1RH	1RA	1LB	12	6
1RB	1LC	1RC	1RH	1LA	0LB	11	6

4.3 Turing machines with 4 states and 2 symbols

- Brady (1964,1965,1966) found a machine M such that $s(M) = 107$ and $\sigma(M) = 13$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simTM42_bb.html`

  Brady conjectured that $S(4,2) = 107$ and $\Sigma(4,2) = 13$.

- Brady (1974,1975) proved this conjecture, and the proof was eventually published in Brady (1983).

- Independently, Machlin and Stout (1990) published another proof of the same result, first reported by Kopp (1981) (Kopp is the maiden name of Machlin).

- Independently, Weimann, Casper and Fenzl (1973) claimed that they proved this conjecture.

1964	Brady	$s = 107$	$\sigma = 13$
1974	Brady	$S(4,2) = 107$	$\Sigma(4,2) = 13$

The winner and some other good machines:

A0	A1	B0	B1	C0	C1	D0	D1	$s(M)$	$\sigma(M)$
1RB	1LB	1LA	0LC	1RH	1LD	1RD	0RA	107	13
1RB	1LD	1LC	0RB	1RA	1LA	1RH	0LC	97	9
1RB	0RC	1LA	1RA	1RH	1RD	1LD	0LB	96	13
1RB	1LB	0LC	0RD	1RH	1LA	1RA	0LA	96	6
1RB	1LD	0LC	0RC	1LC	1LA	1RH	0LA	84	11
1RB	1RH	1LC	0RD	1LA	1LB	0LC	1RD	83	8
1RB	0RD	1LC	0LA	1RA	1LB	1RH	0RC	78	12

4.4 Turing machines with 5 states and 2 symbols

- Green (1964) found a machine M with $\sigma(M) = 17$.

- Lynn (1972) found machines M and N with $s(M) = 435$ and $\sigma(N) = 22$.

- Weimann (1973) found a machine M with $s(M) = 556$ and $\sigma(M) = 40$.
- Lynn, cited by Brady (1983), found in 1974 machines M and N with $s(M) = 7,707$ and $\sigma(N) = 112$.
- Uwe Schult, cited by Ludewig et al. (1983) and by Dewdney (1984a), found, in August 1982, a machine M with $s(M) = 134,467$ and $\sigma(M) = 501$. This machine was analyzed independently by Ludewig (in Ludewig et al. (1983)), by Robinson (cited by Dewdney (1984b)), and by Michel (1993).
- George Uhing, cited by Dewdney (1985a,b), found, in December 1984, a machine M with $s(M) = 2,133,492$ and $\sigma(M) = 1,915$. This machine was analyzed by Michel (1993).
- George Uhing, cited by Brady (1988), found, in February 1986, a machine M with $s(M) = 2,358,064$ (and $\sigma(M) = 1,471$). This machine was analyzed by Michel (1993). Machine 7 in Marxen bb-list, in

  `http://www.drb.insel.de/~heiner/BB/bb-list`

  can be obtained from Uhing's one, as given by Brady (1988), by the permutation of states (A D B E). See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simmbL5_7.html`

- Heiner Marxen and Jürgen Buntrock found, in August 1989, a machine M with $s(M) = 11,798,826$ and $\sigma(M) = 4,098$. This machine was cited by Marxen and Buntrock (1990), and by Machlin and Stout (1990), and was analyzed by Michel (1993). See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simmbL5_2.html`

- Heiner Marxen and Jürgen Buntrock found, in September 1989, a machine M with $s(M) = 23,554,764$ (and $\sigma(M) = 4,097$). This machine was cited by Machlin and Stout (1990), and was analyzed by Michel (1993). See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simmbL5_3.html`

  and analysis by P. Michel in Section 5.2.2.

- Heiner Marxen and Jürgen Buntrock found, in September 1989, a machine M with $s(M) = 47,176,870$ and $\sigma(M) = 4,098$. This machine was cited by Marxen and Buntrock (1990), and was analyzed by Buro (1990) and by Michel (1993). See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simmbL5_1.html`

  analysis by Buro in (p. 64-67)

  `https://skatgame.net/mburo/ps/diploma.pdf`

  and analysis by P. Michel in Section 5.2.1. It is the current record holder.

- Marxen gives a list of machines M with high values of $s(M)$ and $\sigma(M)$ in

  `http://www.drb.insel.de/~heiner/BB/bb-list`

- The study of Turing machines with 5 states and 2 symbols is still going on. Marxen and Buntrock (1990), Skelet, and Hertel (2009) created programs to detect never halting machines, and manually proved that some machines, undetected by their programs, never halt. In each case, about a hundred holdouts were resisting computer and manual analyses. See Skelet's study in

  `http://skelet.ludost.net/bb/index.html`

  The number of holdouts is gradually shrinking, due to the work of many people. See the 42 holdouts of Skelet in

  `http://skelet.ludost.net/bb/nreg.html`

  and the study of 14 of them in

  `http://googology.wikia.com/wiki/Forum:Sigma_project`

- Daniel briggs did some work: see

  `https://web.archive.org/web/20121026023118/\protect\vrule width0pt\protect\href{http://web.m`

- Norbert Bátfai, allowing transitions where the head can stand still, found, in August 2009, a machine M with $s(M) = 70,740,810$ and $\sigma(M) = 4098$. Note that this machine does not follow the current rules of the busy beaver competition. See Bátfai's study in

  `http://arxiv.org/abs/0908.4013`

1964	Green		$\sigma = 17$
1972	Lynn	$s = 435$	$\sigma = 22$
1973	Weimann	$s = 556$	$\sigma = 40$
1974	Lynn	$s = 7{,}707$	$\sigma = 112$
August 1982	Schult	$s = 134{,}467$	$\sigma = 501$
December 1984	Uhing	$s = 2{,}133{,}492$	$\sigma = 1{,}915$
February 1986	Uhing	$s = 2{,}358{,}064$	
February 1990	Marxen, Buntrock	$s = 47{,}176{,}870$	$\sigma = 4{,}098$

The record holder and some other good machines:

A0	A1	B0	B1	C0	C1	D0	D1	E0	E1	$s(M)$	$\sigma(M)$
1RB	1LC	1RC	1RB	1RD	0LE	1LA	1LD	1RH	0LA	47,176,870	4098
1RB	0LD	1LC	1RD	1LA	1LC	1RH	1RE	1RA	0RB	23,554,764	4097
1RB	1RA	1LC	1LB	1RA	0LD	0RB	1LE	1RH	0RB	11,821,234	4097
1RB	1RA	1LC	1LB	1RA	0LD	1RC	1LE	1RH	0RB	11,821,220	4097
1RB	1RA	0LC	0RC	1RH	1RD	1LE	0LA	1LA	1LE	11,821,190	4096
1RB	1RA	1LC	0RD	1LA	1LC	1RH	1RE	1LC	0LA	11,815,076	4096
1RB	1RA	1LC	1LB	1RA	0LD	0RB	1LE	1RH	1LC	11,811,040	4097
1RB	1RA	1LC	1LB	0RC	1LD	1RA	0LE	1RH	1LC	11,811,040	4097
1RB	1RA	1LC	1LB	1RA	0LD	1RC	1LE	1RH	1LC	11,811,026	4097
1RB	1RA	0LC	0RC	1RH	1RD	1LE	1RB	1LA	1LE	11,811,010	4096
1RB	1RA	1LC	1LB	1RA	1LD	0RE	0LE	1RH	1LC	11,804,940	4097
1RB	1RA	1LC	1LB	1RA	1LD	1RA	0LE	1RH	1LC	11,804,926	4097
1RB	1RA	1LC	0RD	1LA	1LC	1RH	1RE	0LE	1RB	11,804,910	4096
1RB	1RA	1LC	0RD	1LA	1LC	1RH	1RE	1LC	1RB	11,804,896	4096
1RB	1RA	1LC	1LB	1RA	1LD	1RA	1LE	1RH	0LC	11,798,826	4098
1RB	1RA	1LC	1RD	1LA	1LC	1RH	0RE	1LC	1RB	11,798,796	4097
1RB	1RA	1LC	1RD	1LA	1LC	1RH	1RE	0LE	0RB	11,792,724	4097
1RB	1RA	1LC	1RD	1LA	1LC	1RH	1RE	1LA	0RB	11,792,696	4097
1RB	1RA	1LC	1RD	1LA	1LC	1RH	1RE	1RA	0RB	11,792,682	4097
0RB	0LC	1RC	1RD	1LA	0LE	1RE	1RH	1LA	1RA	2,358,065	1471
1RB	1RH	1LC	1RC	0RE	0LD	1LC	0LB	1RD	1RA	2,358,064	1471
1RB	1LC	0LA	0LD	1LA	1RH	1LB	1RE	0RD	0RB	2,133,492	1915
1RB	0LC	1RC	1RD	1LA	0RB	0RE	1RH	1LC	1RA	134,467	501

(All these machines can be found in Buro (1990), pp. 69-70. The machines M with $\sigma(M) > 1471$ were discovered by Marxen and Buntrock. The machine with the transition $(A, 0) \to (0, R, B)$ was discovered by Buro, the next two ones were by Uhing, and the last one was by Schult. Heiner Marxen says there are no other σ values within the σ range above).

4.5 Turing machines with 6 states and 2 symbols

- Green (1964) found a machine M with $\sigma(M) = 35$.
- Lynn (1972) found a machine M with $s(M) = 522$ and $\sigma(M) = 42$.
- Brady (1983) found machines M and N with $s(M) = 13,488$ and $\sigma(N) = 117$.
- Uwe Schult, cited by Ludewig et al. (1983) and by Dewdney (1984a), found, in December 1982, a machine M with $s(M) = 4,208,824$ and $\sigma(M) = 2,075$.
- Heiner Marxen and Jürgen Buntrock found, in January 1990, a machine M with $s(M) = 13,122,572,797$ and $\sigma(M) = 136,612$. This machine was cited by Marxen and Buntrock (1990). See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simmbL6_1.html`
- Heiner Marxen and Jürgen Buntrock found, in January 1990, a machine M with $s(M) = 8,690,333,381,690,951$ and $\sigma(M) = 95,524,079$. This machine was posted on the web (Google groups) on September 3, 1997. See machine 2 in Marxen's bb-list in

  `http://www.drb.insel.de/~heiner/BB/bb-list`

See study by H. Marxen in

http://www.drb.insel.de/~heiner/BB/simmbL6_2.html

See analysis by R. Munafo in his website:

http://mrob.com/pub/math/ln-notes1-4.html#mb-bb-1

and in Section 5.3.8.

- Heiner Marxen and Jürgen Buntrock found, in July 2000, a machine M with $s(M) > 5.3 \times 10^{42}$ and $\sigma(M) > 2.5 \times 10^{21}$. This machine was posted on the web (Google groups) on August 5, 2000. See machine 3 in Marxen's bb-list:

  http://www.drb.insel.de/~heiner/BB/bb-list

  and machine k in Marxen's bb-6list:

  http://www.drb.insel.de/~heiner/BB/bb-6list

  See study by H. Marxen in:

  http://www.drb.insel.de/~heiner/BB/simmbL6_3.html

- Heiner Marxen and Jürgen Buntrock found, in August 2000, a machine M with $s(M) > 6.1 \times 10^{119}$ and $\sigma(M) > 1.4 \times 10^{60}$. This machine was posted on the web (Google groups) on October 23, 2000. See machine o in Marxen's bb-6list in:

  http://www.drb.insel.de/~heiner/BB/bb-6list

  See study by H. Marxen in:

  http://www.drb.insel.de/~heiner/BB/simmbL6_o.html

  See analysis by P. Michel in Section 5.3.7.

- Heiner Marxen and Jürgen Buntrock found, in August 2000, a machine M with $s(M) > 6.1 \times 10^{925}$ and $\sigma(M) > 6.4 \times 10^{462}$. This machine was posted on the web (Google groups) on October 23, 2000. See machine q in Marxen's bb-6list in

  http://www.drb.insel.de/~heiner/BB/bb-6list

  See study by Marxen in

  http://www.drb.insel.de/~heiner/BB/simmbL6_q.html

  See analyses by R. Munafo, the short one in

  http://mrob.com/pub/math/ln-notes1-5.html#mb6q

  or the long one in

  http://mrob.com/pub/math/ln-mb6q.html

  and see analysis by P. Michel in Section 5.3.6.

- Heiner Marxen and Jürgen Buntrock found, in February 2001, a machine M with $s(M) > 3.0 \times 10^{1730}$ and $\sigma(M) > 1.2 \times 10^{865}$. This machine was posted on the web (Google groups) on March 5, 2001. See machine r in Marxen's bb-6list in

  http://www.drb.insel.de/~heiner/BB/bb-6list

  See study by Marxen in

`http://www.drb.insel.de/~heiner/BB/simmbL6_r.html`

See analysis by P. Michel in Section 5.3.5.

- Terry and Shawn Ligocki found, in November 2007, a machine M with $s(M) > 8.9 \times 10^{1762}$ and $\sigma(M) > 2.5 \times 10^{881}$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig62_a.html`

  See analysis by P. Michel in Section 5.3.4.

- Terry and Shawn Ligocki found, in December 2007, a machine M with $s(M) > 2.5 \times 10^{2879}$ and $\sigma(M) > 4.6 \times 10^{1439}$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig62_b.html`

  See analysis by P. Michel in Section 5.3.3.

- Pavel Kropitz found, in May 2010, a machine M with $s(M) > 3.8 \times 10^{21132}$ and $\sigma(M) > 3.1 \times 10^{10566}$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simKro62_a.html`

  See analysis in Section 5.3.2.

- Pavel Kropitz found, in June 2010, a machine M with $s(M) > 7.4 \times 10^{36534}$ and $\sigma(M) > 3.5 \times 10^{18267}$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simKro62_b.html`

  See analysis by P. Michel in Section 5.3.1. It is the current record holder.

- Marxen gives a list of machines M with high values of $s(M)$ and $\sigma(M)$ in

  `http://www.drb.insel.de/~heiner/BB/bb-6list`

1964	Green		$\sigma = 35$
1972	Lynn	$s = 522$	$\sigma = 42$
1983	Brady	$s = 13{,}488$	$\sigma = 117$
December 1982	Schult	$s = 4{,}208{,}824$	$\sigma = 2{,}075$
February 1990	Marxen, Buntrock	$s = 13{,}122{,}572{,}797$	$\sigma = 136{,}612$
September 1997	Marxen, Buntrock	$s = 8{,}690{,}333{,}381{,}690{,}951$	$\sigma = 95{,}524{,}079$
August 2000	Marxen, Buntrock	$s > 5.3 \times 10^{42}$	$\sigma > 2.5 \times 10^{21}$
October 2000	Marxen, Buntrock	$s > 6.1 \times 10^{925}$	$\sigma > 6.4 \times 10^{462}$
March 2001	Marxen, Buntrock	$s > 3.0 \times 10^{1730}$	$\sigma > 1.2 \times 10^{865}$
November 2007	T. and S. Ligocki	$s > 8.9 \times 10^{1762}$	$\sigma > 2.5 \times 10^{881}$
December 2007	T. and S. Ligocki	$s > 2.5 \times 10^{2879}$	$\sigma > 4.6 \times 10^{1439}$
May 2010	Kropitz	$s > 3.8 \times 10^{21132}$	$\sigma > 3.1 \times 10^{10566}$
June 2010	Kropitz	$s > 7.4 \times 10^{36534}$	$\sigma > 3.5 \times 10^{18267}$

The record holder and some other good machines:

A0	A1	B0	B1	C0	C1	D0	D1	E0	E1	F0	F1	$s(M) >$	$\sigma(M) >$
1RB	1LE	1RC	1RF	1LD	0RB	1RE	0LC	1LA	0RD	1RH	1RC	7.4×10^{36534}	3.5×10^{18267}
1RB	0LD	1RC	0RF	1LC	1LA	0LE	1RH	1LA	0RB	0RC	0RE	3.8×10^{21132}	3.1×10^{10566}
1RB	0LE	1LC	0RA	1LD	0RC	1LE	0LF	1LA	1LC	1LE	1RH	2.5×10^{2879}	4.6×10^{1439}
1RB	0RF	0LB	1LC	1LD	0RC	1LE	1RH	1LF	0LD	1RA	0LE	8.9×10^{1762}	2.5×10^{881}
1RB	0LF	0RC	0RD	1LD	1RE	0LE	0LD	0RA	1RC	1LA	1RH	3.0×10^{1730}	1.2×10^{865}
1RB	0LB	0RC	1LB	1RD	0LA	1LE	1LF	1LA	0LD	1RH	1LE	6.1×10^{925}	6.4×10^{462}
1RB	0LC	1LA	1RC	1RA	0LD	1LE	1LC	1RF	1RH	1RA	1RE	6.1×10^{119}	1.4×10^{60}
1RB	0LB	1LC	0RE	1RE	0LD	1LA	1LA	0RA	0RF	1RE	1RH	5.5×10^{99}	6.9×10^{49}
1RB	0LC	1LA	1LD	1RD	0RC	0LB	0RE	1RC	1LF	1LE	1RH	3.2×10^{98}	1.1×10^{49}
1RB	0LC	1LA	1RD	1RA	0LE	1RA	0RB	1LF	1LC	1RD	1RH	2.0×10^{95}	6.7×10^{47}
1RB	0LC	1LA	1RD	0LB	0LE	1RA	0RB	1LF	1LC	1RD	1RH	2.0×10^{95}	6.7×10^{47}
1RB	0RC	0LA	0RD	1RD	1RH	1LE	0LD	1RF	1LB	1RA	1RE	5.3×10^{42}	2.5×10^{21}

4.6 Turing machines with 7 states and 2 symbols

- Green (1964) found a machine M with $\sigma(M) = 22,961$.
- This machine was superseded by the machine with 6 states and 2 symbols found in January 1990 by Heiner Marxen and Jürgen Buntrock.
- "Wythagoras" found, in March 2014, a machine M with $s(M) > \sigma(M) > 10^{10^{10^{10^{18,705,352}}}}$. This machine comes from the (6,2)-TM found by Pavel Kropitz in June 2010, as follows: A seventh state G is added, with the transition (G,0) $\rightarrow$ (1,L,E). This state G becomes the initial state. Then the machine is normalized by swapping Left and Right and by the circular permutation of states (A C F E B D G). See

  `http://googology.wikia.com/wiki/User_blog:Wythagoras/A_good_bound_for_S(7)%3F`

1964	Green	$\sigma = 22{,}961$
1990	Marxen, Buntrock	superseded by a (6,2)-TM
March 2014	"Wythagoras"	$s > \sigma > 10^{10^{10^{10^{18,705,352}}}}$

The record holder:

A0	A1	B0	B1	C0	C1	D0	D1	E0	E1	F0	F1	G0	G1
1RB		1RC	0LG	1LD	1RB	1LF	1LE	1RH	1LF	1RG	0LD	1LB	0RF

4.7 Turing machines with 2 states and 3 symbols

- Brady (1988) found a machine M with $s(M) = 38$ See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simTM23_cb.html`

- This machine was found independently by Michel (2004), who gave $\sigma(M) = 9$ and conjectured that $S(2,3) = 38$ and $\Sigma(2,3) = 9$.
- Lafitte and Papazian (2007) proved this conjecture. T. and S. Ligocki (unpublished) proved this conjecture, independently.

1988	Brady	$s = 38$	$\sigma = 9$
2007	Lafitte, Papazian	$S(2,3) = 38$	$\Sigma(2,3) = 9$

The winner and some other good machines:

A0	A1	A2	B0	B1	B2	$s(M)$	$\sigma(M)$
1RB	2LB	1RH	2LA	2RB	1LB	38	9
1RB	0LB	1RH	2LA	1RB	1RA	29	8
0RB	2LB	1RH	1LA	1RB	1RA	27	6
1RB	2LA	1RH	1LB	1LA	0RA	26	6
1RB	1LA	1LB	0LA	2RA	1RH	26	6
1RB	1LB	1RH	2LA	2RB	1LB	24	7

4.8 Turing machines with 3 states and 3 symbols

- Korfhage (1966) (p. 114) claimed that $S(3,3) \geq 57$ and $\Sigma(3,3) \geq 12$. He did not give a machine. He gave a list of authors (C.Y. Lee, Tibor Rado, Shen Lin, Patrick Fischer, Milton Green and David Jefferson) without specifying who found this result. Note that the definition of $\Sigma(3,3)$ used in this book could be different from the current definition (i.e., number of non-blank symbols).

- Michel (2004) found machines M and N with $s(M) = 40,737$ and $\sigma(N) = 208$.

- Brady found, in November 2004, a machine M with $s(M) = 29,403,894$ and $\sigma(M) = 5600$. See `http://www.cse.unr.edu/~al/BusyBeaver.html`

  See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simAB3Y_b.html`

- Brady found, in December 2004, a machine M with $s(M) = 92,649,163$ and $\sigma(M) = 13,949$. See `http://www.cse.unr.edu/~al/BusyBeaver.html`

  See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simAB3Y_c.html`

  See analysis by P. Michel in Section 5.4.8.

- Myron P. Souris found, in July 2005 (M.P. Souris said: actually in 1995, but then no one seemed to care), machines M and N with $s(M) = 544,884,219$ and $\sigma(N) = 36,089$. See study of M by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simMS33_b.html`

  and study of N by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simMS33_a.html`

  See analysis of M by P. Michel in Section 5.4.6,

  and analysis of N by P. Michel in Section 5.4.7.

- Grégory Lafitte and Christophe Papazian found, in August 2005, a machine M with $s(M) = 4,939,345,068$ and $\sigma(M) = 107,900$. Eee study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLaf33_b.html`

  See analysis by P. Michel in Section 5.4.5.

- Grégory Lafitte and Christophe Papazian found, in September 2005, a machine M with $s(M) = 987,522,842,126$ and $\sigma(M) = 1,525,688$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLaf33_d.html`

  See analysis by P. Michel in Section 5.4.4.

- Grégory Lafitte and Christophe Papazian found, in April 2006, a machine M with $s(M) = 4,144,465,135,614$ and $\sigma(M) = 2,950,149$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLaf33_e.html`

  See analysis by P. Michel in Section 5.4.3.

- Terry and Shawn Ligocki found, in August 2006, a machine M with $s(M) = 4,345,166,620,336,565$ and $\sigma(M) = 95,524,079$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig33_a.html`

  See analysis in Section 5.4.2.

- Terry and Shawn Ligocki found, in November 2007, a machine M with $s(M) = 119,112,334,170,342,540$ and $\sigma(M) = 374,676,383$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig33_b.html`

  See analysis by P. Michel in Section 5.4.1.

  It is the current record holder.

- Brady gives a list of machines with high values of $s(M)$ in

  `http://www.cse.unr.edu/~al/BusyBeaver.html`

1966	cited by Korfhage	$s = 57$	$\sigma' = 12$
October 2004	Michel	$s = 40{,}737$	$\sigma = 208$
November 2004	Brady	$s = 29{,}403{,}894$	$\sigma = 5{,}600$
December 2004	Brady	$s = 92{,}649{,}163$	$\sigma = 13{,}949$
July 2005	Souris	$s = 544{,}884{,}219$	$\sigma = 36{,}089$
August 2005	Lafitte, Papazian	$s = 4{,}939{,}345{,}068$	$\sigma = 107{,}900$
September 2005	Lafitte, Papazian	$s = 987{,}522{,}842{,}126$	$\sigma = 1{,}525{,}688$
April 2006	Lafitte, Papazian	$s = 4{,}144{,}465{,}135{,}614$	$\sigma = 2{,}950{,}149$
August 2006	T. and S. Ligocki	$s = 4{,}345{,}166{,}620{,}336{,}565$	$\sigma = 95{,}524{,}079$
November 2007	T. and S. Ligocki	$s = 119{,}112{,}334{,}170{,}342{,}540$	$\sigma = 374{,}676{,}383$

The record holder and some other good machines:

A0	A1	A2	B0	B1	B2	C0	C1	C2	$s(M)$	$\sigma(M)$
1RB	2LA	1LC	0LA	2RB	1LB	1RH	1RA	1RC	119,112,334,170,342,540	374,676,383
1RB	2RC	1LA	2LA	1RB	1RH	2RB	2RA	1LC	4,345,166,620,336,565	95,524,079
1RB	1RH	2LC	1LC	2RB	1LB	1LA	2RC	2LA	4,144,465,135,614	2,950,149
1RB	2LA	1RA	1RC	2RB	0RC	1LA	1RH	1LA	987,522,842,126	1,525,688
1RB	1RH	2RB	1LC	0LB	1RA	1RA	2LC	1RC	4,939,345,068	107,900
1RB	2LA	1RA	1LB	1LA	2RC	1RH	1LC	2RB	1,808,669,066	43,925
1RB	2LA	1RA	1LC	1LA	2RC	1RH	1LA	2RB	1,808,669,046	43,925
1RB	1LB	2LA	1LA	1RC	1RH	0LA	2RC	1LC	544,884,219	32,213
1RB	0LA	1LA	2RC	1RC	1RH	2LC	1RA	0RC	408,114,977	20,240
1RB	2RA	2RC	1LC	1RH	1LA	1RA	2LB	1LC	310,341,163	36,089
1RB	1RH	2LC	1LC	2RB	1LB	1LA	0RB	2LA	92,649,163	13,949
1RB	2LA	1LA	2LA	1RC	2RB	1RH	0LC	0RA	51,525,774	7,205
1RB	2RA	1LA	2LA	2LB	2RC	1RH	2RB	1RB	47,287,015	12,290
1RB	2RA	1LA	2LC	0RC	1RB	1RH	2LA	1RB	29,403,894	5,600

(The first two machines were discovered by Terry and Shawn Ligocki, the next five ones were by Lafitte and Papazian, the next three ones were by Souris, and the last four ones were by Brady).

4.9 Turing machines with 4 states and 3 symbols

- Terry and Shawn Ligocki found, in April 2005, a machine M with $s(M) = 250,096,776$ and $\sigma(M) = 15,008$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig43_a.html`

- This machine was superseded by the machines with 3 states and 3 symbols found in July 2005 by Myron P. Souris.

- Terry and Shawn Ligocki found, in October 2007, a machine M with $s(M) > 1.5 \times 10^{1426}$ and $\sigma(M) > 1.1 \times 10^{713}$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig43_b.html`

- Terry and Shawn Ligocki found successively, in November 2007, machines M with

  - $s(M) > 7.7 \times 10^{1618}$ and $\sigma(M) > 1.6 \times 10^{809}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig43_c.html`
  - $s(M) > 3.7 \times 10^{1973}$ and $\sigma(M) > 8.0 \times 10^{986}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig43_d.html`
  - $s(M) > 3.9 \times 10^{7721}$ and $\sigma(M) > 4.0 \times 10^{3860}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig43_e.html`
  - $s(M) > 3.9 \times 10^{9122}$ and $\sigma(M) > 2.5 \times 10^{4561}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig43_f.html`

- Terry and Shawn Ligocki found successively, in December 2007, machines M with

  - $s(M) > 7.9 \times 10^{9863}$ and $\sigma(M) > 8.9 \times 10^{4931}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig43_g.html`
  - $s(M) > 5.3 \times 10^{12068}$ and $\sigma(M) > 4.2 \times 10^{6034}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig43_h.html`

- Terry and Shawn Ligocki found, in January 2008, a machine M with $s(M) > 1.0 \times 10^{14072}$ and $\sigma(M) > 1.3 \times 10^{7036}$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig43_i.html`

  It is the current record holder.

April 2005	T. and S. Ligocki	$s = 250{,}096{,}776$	$\sigma = 15{,}008$
July 2005	Souris	superseded by a (3,3)-TM	
October 2007	T. and S. Ligocki	$s > 1.5 \times 10^{1426}$	$\sigma > 1.1 \times 10^{713}$
November 2007	T. and S. Ligocki	$s > 7.7 \times 10^{1618}$ $s > 3.7 \times 10^{1973}$ $s > 3.9 \times 10^{7721}$ $s > 3.9 \times 10^{9122}$	$\sigma > 1.6 \times 10^{809}$ $\sigma > 8.0 \times 10^{986}$ $\sigma > 4.0 \times 10^{3860}$ $\sigma > 2.5 \times 10^{4561}$
December 2007	T. and S. Ligocki	$s > 7.9 \times 10^{9863}$ $s > 5.3 \times 10^{12068}$	$\sigma > 8.9 \times 10^{4931}$ $\sigma > 4.2 \times 10^{6034}$
January 2008	T. and S. Ligocki	$s > 1.0 \times 10^{14072}$	$\sigma > 1.3 \times 10^{7036}$

The record holder and the past record holders:

A0	A1	A2	B0	B1	B2	C0	C1	C2	D0	D1	D2	$s(M)$	$\sigma(M)$
1RB	1RH	2RC	2LC	2RD	0LC	1RA	2RB	0LB	1LB	0LD	2RC	$> 1.0 \times 10^{14072}$	$> 1.3 \times 10^{7036}$
1RB	0LB	1RD	2RC	2LA	0LA	1LB	0LA	0LA	1RA	0RA	1RH	$> 5.3 \times 10^{12068}$	$> 4.2 \times 10^{6034}$
1RB	1LD	1RH	1RC	2LB	2LD	1LC	2RA	0RD	1RC	1LA	0LA	$> 7.9 \times 10^{9863}$	$> 8.9 \times 10^{4931}$
1RB	2LD	1RH	2LC	2RC	2RB	1LD	0RC	1RC	2LA	2LD	0LB	$> 3.9 \times 10^{9122}$	$> 2.5 \times 10^{4561}$
1RB	1LA	1RD	2LC	0RA	1LB	2LA	0LB	0RD	2RC	1RH	0LC	$> 3.9 \times 10^{7721}$	$> 4.0 \times 10^{3860}$
1RB	1RA	0LB	2LC	1LB	1RC	0RD	2LC	1RA	2RA	1RH	1RC	$> 3.7 \times 10^{1973}$	$> 8.0 \times 10^{986}$
1RB	2RC	1RA	2LC	1LA	1LB	2LD	0LB	0RC	0RD	1RH	0RA	$> 7.7 \times 10^{1618}$	$> 1.6 \times 10^{809}$
1RB	0LC	1RH	2LC	1RD	0LB	2LA	1LC	1LA	1RB	2LD	2RA	$> 1.5 \times 10^{1426}$	$> 1.1 \times 10^{713}$
0RB	1LD	1RH	1LA	1RC	1RD	1RB	2LC	1RC	1LA	1LC	2RB	250,096,776	15,008

4.10 Turing machines with 2 states and 4 symbols

- Brady (1988) found a machine M with $s(M) = 7,195$.

- This machine was found independently and analyzed by Michel (2004), who gave $\sigma(M) = 90$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simTM24_b.html`

  See analysis by P. Michel in Section 5.5.2.

- Terry and Shawn Ligocki found, in February 2005, a machine M with $s(M) = 3,932,964$ and $\sigma(M) = 2,050$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig24_a.html`

  See analysis by P. Michel in Section 5.5.1.

  It is the current record holder. There is no machine between the first two ones (Ligocki, Brady). There is no machine such that $3,932,964 < s(M) < 200,000,000$ (Ligocki, September 2005).

1988	Brady	$s = 7{,}195$	$\sigma = 90$
February 2005	T. and S. Ligocki	$s = 3{,}932{,}964$	$\sigma = 2{,}050$

The record holder and some other good machines:

A0	A1	A2	A3	B0	B1	B2	B3	$s(M)$	$\sigma(M)$
1RB	2LA	1RA	1RA	1LB	1LA	3RB	1RH	3,932,964	2,050
1RB	3LA	1LA	1RA	2LA	1RH	3RA	3RB	7,195	90
1RB	3LA	1LA	1RA	2LA	1RH	3LA	3RB	6,445	84
1RB	3LA	1LA	1RA	2LA	1RH	2RA	3RB	6,445	84
1RB	2RB	3LA	2RA	1LA	3RB	1RH	1LB	2,351	60

4.11 Turing machines with 3 states and 4 symbols

- Terry and Shawn Ligocki found, in April 2005, a machine M with $s(M) = 262,759,288$ and $\sigma(M) = 17,323$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig34_a.html`

- This machine was superseded by the machines with 3 states and 3 symbols found in July 2005 by Myron P. Souris.

- Terry and Shawn Ligocki found, in September 2007, a machine M with $s(M) > 5.7 \times 10^{52}$ and $\sigma(M) > 2.4 \times 10^{26}$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig34_b.html`

- Terry and Shawn Ligocki found successively, in October 2007, machines M with

  - $s(M) > 4.3 \times 10^{281}$ and $\sigma(M) > 6.0 \times 10^{140}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig34_c.html`
  - $s(M) > 7.6 \times 10^{868}$ and $\sigma(M) > 4.6 \times 10^{434}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig34_d.html`
  - $s(M) > 3.1 \times 10^{1256}$ and $\sigma(M) > 2.1 \times 10^{628}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig34_e.html`

- Terry and Shawn Ligocki found successively, in November 2007, machines M with

  - $s(M) > 8.4 \times 10^{2601}$ and $\sigma(M) > 1.7 \times 10^{1301}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig34_f.html`
  - $s(M) > 3.4 \times 10^{4710}$ and $\sigma(M) > 1.4 \times 10^{2355}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig34_g.html`
  - $s(M) > 5.9 \times 10^{4744}$ and $\sigma(M) > 2.2 \times 10^{2372}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig34_h.html`

- Terry and Shawn Ligocki found, in December 2007, a machine M with $s(M) > 5.2 \times 10^{13036}$ and $\sigma(M) > 3.7 \times 10^{6518}$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig34_i.html`

  It is the current record holder.

April 2005	T. and S. Ligocki	$s = 262{,}759{,}288$	$\sigma = 17{,}323$
July 2005	Souris	superseded by a (3,3)-TM	
September 2007	T. and S. Ligocki	$s > 5.7 \times 10^{52}$	$\sigma > 2.4 \times 10^{26}$
October 2007	T. and S. Ligocki	$s > 4.3 \times 10^{281}$ $s > 7.6 \times 10^{868}$ $s > 3.1 \times 10^{1256}$	$\sigma > 6.0 \times 10^{140}$ $\sigma > 4.6 \times 10^{434}$ $\sigma > 2.1 \times 10^{628}$
November 2007	T. and S. Ligocki	$s > 8.4 \times 10^{2601}$ $s > 3.4 \times 10^{4710}$ $s > 5.9 \times 10^{4744}$	$\sigma > 1.7 \times 10^{1301}$ $\sigma > 1.4 \times 10^{2355}$ $\sigma > 2.2 \times 10^{2372}$
December 2007	T. and S. Ligocki	$s > 5.2 \times 10^{13036}$	$\sigma > 3.7 \times 10^{6518}$

The record holder and the past record holders:

A0	A1	A2	A3	B0	B1	B2	B3	C0	C1	C2	C3	$s(M)$	$\sigma(M)$
1RB	1RA	2LB	3LA	2LA	0LB	1LC	1LB	3RB	3RC	1RH	1LC	$> 5.2 \times 10^{13036}$	$> 3.7 \times 10^{6518}$
1RB	1RA	1LB	1RC	2LA	0LB	3LC	1RH	1LB	0RC	2RA	2RC	$> 5.9 \times 10^{4744}$	$> 2.2 \times 10^{2372}$
1RB	2LB	2RA	1LA	2LA	1RC	0LB	2RA	1RB	3LC	1LA	1RH	$> 3.4 \times 10^{4710}$	$> 1.4 \times 10^{2355}$
1RB	1LA	3LA	3RC	2LC	2LB	1RB	1RA	2LA	3LC	1RH	1LB	$> 8.4 \times 10^{2601}$	$> 1.7 \times 10^{1301}$
1RB	3LA	3RC	1RA	2RC	1LA	1RH	2RB	1LC	1RB	1LB	2RA	$> 3.1 \times 10^{1256}$	$> 2.1 \times 10^{628}$
1RB	0RB	3LC	1RC	0RC	1RH	2RC	3RC	1LB	2LA	3LA	2RB	$> 7.6 \times 10^{868}$	$> 4.6 \times 10^{434}$
1RB	3RB	2LC	3LA	0RC	1RH	2RC	1LB	1LB	2LA	3RC	2LC	$> 4.3 \times 10^{281}$	$> 6.0 \times 10^{140}$
1RB	1LA	1LB	1RA	0LA	2RB	2LC	1RH	3RB	2LB	1RC	0RC	$> 5.7 \times 10^{52}$	$> 2.4 \times 10^{26}$
1RB	3LC	0RA	0LC	2LC	3RC	0RC	1LB	1RA	0LB	0RB	1RH	262,759,288	17,323

4.12 Turing machines with 2 states and 5 symbols

- Terry and Shawn Ligocki found, in February 2005, machines M and N with $s(M) = 16,268,767$ and $\sigma(N) = 4,099$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig25_a.html`

- Terry and Shawn Ligocki found, in April 2005, a machine M with $s(M) = 148,304,214$ and $\sigma(M) = 11,120$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig25_c.html`

- Grégory Lafitte and Christophe Papazian found, in August 2005, a machine M with $s(M) = 8,619,024,596$ and $\sigma(M) = 90,604$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLaf25_a.html`

- Grégory Lafitte and Christophe Papazian found, in September 2005, a machine M with $\sigma(M) = 97,104$ (and $s(M) = 7,543,673,517$). See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLaf25_c.html`

- Grégory Lafitte and Christophe Papazian found, in October 2005, a machine M with $s(M) = 233,431,192,481$ and $\sigma(M) = 458,357$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLaf25_d.html`

- Grégory Lafitte and Christophe Papazian found, in October 2005, a machine M with $s(M) = 912,594,733,606$ and $\sigma(M) = 1,957,771$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLaf25_f.html`

  See analysis by P. Michel in Section 5.6.6.

- Grégory Lafitte and Christophe Papazian found, in December 2005, a machine M with $s(M) = 924,180,005,181$ (and $\sigma(M) = 1,137,477$). See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLaf25_g.html`

  See analysis by P. Michel in Section 5.6.5.

- Grégory Lafitte and Christophe Papazian found, in May 2006, a machine M with $s(M) = 3,793,261,759,791$ and $\sigma(M) = 2,576,467$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLaf25_h.html`

  See analysis by P. Michel in Section 5.6.4.

- Grégory Lafitte and Christophe Papazian found, in June 2006, a machine M with $s(M) = 14,103,258,269,249$ and $\sigma(M) = 4,848,239$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLaf25_i.html`

  See analysis by P. Michel in Section 5.6.3.

- Grégory Lafitte and Christophe Papazian found, in July 2006, a machine M with $s(M) = 26,375,397,569,930$ (and $\sigma(M) = 143$). See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLaf25_j.html`

  See comments in Section 5.8.

- Terry and Shawn Ligocki found, in August 2006, a machine M with

  $$s(M) = 7,069,449,877,176,007,352,687$$

  and $\sigma(M) = 172,312,766,455$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig25_j.html`

  See analysis in Section 5.6.2.

- Terry and Shawn Ligocki found, in October 2007, a machine M with $s(M) > 5.2 \times 10^{61}$ and $\sigma(M) > 9.3 \times 10^{30}$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig25_k.html`

- Terry and Shawn Ligocki found, in October 2007, two machines M and N with $s(M) = s(N) > 1.6 \times 10^{211}$ and $\sigma(M) = \sigma(N) > 5.2 \times 10^{105}$. See study by H. Marxen of M in

  `http://www.drb.insel.de/~heiner/BB/simLig25_l.html`

  and study by H. Marxen of N in

  `http://www.drb.insel.de/~heiner/BB/simLig25_m.html`

- Terry and Shawn Ligocki found, in November 2007, a machine M with $s(M) > 1.9 \times 10^{704}$ and $\sigma(M) > 1.7 \times 10^{352}$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig25_n.html`

  See analysis by P. Michel in Section 5.6.1.

  It is the current record holder.

February 2005	T. and S. Ligocki	$s = 16{,}268{,}767$	$\sigma = 4{,}099$
April 2005	T. and S. Ligocki	$s = 148{,}304{,}214$	$\sigma = 11{,}120$
August 2005	Lafitte, Papazian	$s = 8{,}619{,}024{,}596$	$\sigma = 90{,}604$
September 2005	Lafitte, Papazian		$\sigma = 97{,}104$
October 2005	Lafitte, Papazian	$s = 233{,}431{,}192{,}481$ $s = 912{,}594{,}733{,}606$	$\sigma = 458{,}357$ $\sigma = 1{,}957{,}771$
December 2005	Lafitte, Papazian	$s = 924{,}180{,}005{,}181$	
May 2006	Lafitte, Papazian	$s = 3{,}793{,}261{,}759{,}791$	$\sigma = 2{,}576{,}467$
June 2006	Lafitte, Papazian	$s = 14{,}103{,}258{,}269{,}249$	$\sigma = 4{,}848{,}239$
July 2006	Lafitte, Papazian	$s = 26{,}375{,}397{,}569{,}930$	
August 2006	T. and S. Ligocki	$s > 7.0 \times 10^{21}$	$\sigma = 172{,}312{,}766{,}455$
October 2007	T. and S. Ligocki	$s > 5.2 \times 10^{61}$ $s > 1.6 \times 10^{211}$	$\sigma > 9.3 \times 10^{30}$ $\sigma > 5.2 \times 10^{105}$
November 2007	T. and S. Ligocki	$s > 1.9 \times 10^{704}$	$\sigma > 1.7 \times 10^{352}$

Note: Two machines were discovered by T. and S. Ligocki in February 2005 with $s(M) = 16,268,767$, and two were in October 2007 with $s(M) > 1.6 \times 10^{211}$.

The record holder and some other good machines:

A0	A1	A2	A3	A4	B0	B1	B2	B3	B4	$s(M)$	$\sigma(M)$
1RB	2LA	1RA	2LB	2LA	0LA	2RB	3RB	4RA	1RH	$> 1.9 \times 10^{704}$	$> 1.7 \times 10^{352}$
1RB	2LA	4RA	2LB	2LA	0LA	2RB	3RB	4RA	1RH	$> 1.6 \times 10^{211}$	$> 5.2 \times 10^{105}$
1RB	2LA	4RA	2LB	2LA	0LA	2RB	3RB	1RA	1RH	$> 1.6 \times 10^{211}$	$> 5.2 \times 10^{105}$
1RB	2LA	4RA	1LB	2LA	0LA	2RB	3RB	2RA	1RH	$> 5.2 \times 10^{61}$	$> 9.3 \times 10^{30}$
1RB	0RB	4RA	2LB	2LA	2LA	1LB	3RB	4RA	1RH	$> 7.0 \times 10^{21}$	172,312,766,455
1RB	3LA	3LB	0LB	1RA	2LA	4LB	4LA	1RA	1RH	339,466,124,499,007,251	1,194,050,967
1RB	3RB	3RA	1RH	2LB	2LA	4RA	4RB	2LB	0RA	339,466,124,499,007,214	1,194,050,967
1RB	1RH	4LA	4LB	2RA	2LB	2RB	3RB	2RA	0RB	91,791,666,497,368,316	620,906,587
1RB	3LA	1LA	0LB	1RA	2LA	4LB	4LA	1RA	1RH	37,716,251,406,088,468	398,005,342
1RB	2RA	1LA	3LA	2RA	2LA	3RB	4LA	1LB	1RH	9,392,084,729,807,219	114,668,733
1RB	2RA	1LA	1LB	3LB	2LA	3RB	1RH	4RA	1LA	417,310,842,648,366	36,543,045

(These machines were discovered by Terry and Shawn Ligocki).

Previous record holders and some other good machines:

A0	A1	A2	A3	A4	B0	B1	B2	B3	B4	$s(M)$	$\sigma(M)$
1RB	3LA	1LA	4LA	1RA	2LB	2RA	1RH	0RA	0RB	26,375,397,569,930	143
1RB	3LB	4LB	4LA	2RA	2LA	1RH	3RB	4RA	3RB	14,103,258,269,249	4,848,239
1RB	3RA	4LB	2RA	3LA	2LA	1RH	4RB	4RB	2LB	3,793,261,759,791	2,576,467
1RB	3RA	1LA	1LB	3LB	2LA	4LB	3RA	2RB	1RH	924,180,005,181	1,137,477
1RB	3LB	1RH	1LA	1LA	2LA	3RB	4LB	4LB	3RA	912,594,733,606	1,957,771
1RB	2RB	3LA	2RA	3RA	2LB	2LA	3LA	4RB	1RH	469,121,946,086	668,420
1RB	3RB	3RB	1LA	3LB	2LA	3RA	4LB	2RA	1RH	233,431,192,481	458,357
1RB	3LA	1LB	1RA	3RA	2LB	3LA	3RA	4RB	1RH	8,619,024,596	90,604
1RB	2RB	3RB	4LA	3RA	0LA	4RB	1RH	0RB	1LB	7,543,673,517	97,104
1RB	4LA	1LA	1RH	2RB	2LB	3LA	1LB	2RA	0RB	7,021,292,621	37
1RB	2RB	3LA	2RA	3RA	2LB	2LA	1LA	4RB	1RH	4,561,535,055	64,665
1RB	3LA	4LA	1RA	1LA	2LA	1RH	4RA	3RB	1RA	148,304,214	11,120
1RB	3LA	4LA	1RA	1LA	2LA	1RH	1LA	3RB	1RA	16,268,767	3,685
1RB	3RB	2LA	0RB	1RH	2LA	4RB	3LB	2RB	3RB	15,754,273	4,099

(The first eleven machines were discovered by Lafitte and Papazian, and the last three ones were by T. and S. Ligocki).

4.13 Turing machines with 2 states and 6 symbols

- Terry and Shawn Ligocki found, in February 2005, machines M and N with $s(M) = 98,364,599$ and $\sigma(N) = 10,574$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig26_a.html`

- Terry and Shawn Ligocki found, in April 2005, a machine M with $s(M) = 493,600,387$ and $\sigma(M) = 15,828$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig26_c.html`

- This machine was superseded by the machine with 2 states and 5 symbols found in August 2005 by Grégory Lafitte and Christophe Papazian.

- Terry and Shawn Ligocki found, in September 2007, a machine M with $s(M) > 2.3 \times 10^{54}$ and $\sigma(M) > 1.9 \times 10^{27}$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig26_d.html`

- This machine was superseded by the machine with 2 states and 5 symbols found in October 2007 by Terry and Shawn Ligocki.

- Terry and Shawn Ligocki found successively, in November 2007, machines M with
  - $s(M) > 4.9 \times 10^{1643}$ and $\sigma(M) > 8.6 \times 10^{821}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig26_e.html`
  - $s(M) > 2.5 \times 10^{9863}$ and $\sigma(M) > 6.9 \times 10^{4931}$. See study by H. Marxen in `http://www.drb.insel.de/~heiner/BB/simLig26_f.html`

- Terry and Shawn Ligocki found, in January 2008, a machine M with $s(M) > 2.4\times10^{9866}$ and $\sigma(M) > 1.9 \times 10^{4933}$. See study by H. Marxen in

  `http://www.drb.insel.de/~heiner/BB/simLig26_g.html`

  It is the current record holder.

February 2005	T. and S. Ligocki	$s = 98,364,599$	$\sigma = 10,574$
April 2005	T. and S. Ligocki	$s = 493,600,387$	$\sigma = 15,828$
August 2005	Lafitte, Papazian	superseded by a (2,5)-TM	
September 2007	T. and S. Ligocki	$s > 2.3 \times 10^{54}$	$\sigma > 1.9 \times 10^{27}$
October 2007	T. and S. Ligocki	superseded by a (2,5)-TM	
November 2007	T. and S. Ligocki	$s > 4.9 \times 10^{1643}$ $s > 2.5 \times 10^{9863}$	$\sigma > 8.6 \times 10^{821}$ $\sigma > 6.9 \times 10^{4931}$
January 2008	T. and S. Ligocki	$s > 2.4 \times 10^{9866}$	$\sigma > 1.9 \times 10^{4933}$

The record holder and the past record holders:

A0	A1	A2	A3	A4	A5	B0	B1	B2	B3	B4	B5	$s(M)$	$\sigma(M)$
1RB	2LA	1RH	5LB	5LA	4LB	1LA	4RB	3RB	5LB	1LB	4RA	$> 2.4 \times 10^{9866}$	$> 1.9 \times 10^{4933}$
1RB	1LB	3RA	4LA	2LA	4LB	2LA	2RB	3LB	1LA	5RA	1RH	$> 2.5 \times 10^{9863}$	$> 6.9 \times 10^{4931}$
1RB	2LB	4RB	1LA	1RB	1RH	1LA	3RA	5RA	4LB	0RA	4LA	$> 4.9 \times 10^{1643}$	$> 8.6 \times 10^{821}$
1RB	0RB	3LA	5LA	1RH	4LB	1LA	2RB	3LA	4LB	3RB	3RA	$> 2.3 \times 10^{54}$	$> 1.9 \times 10^{27}$
1RB	2LA	1RA	1RA	5LB	4LB	1LB	1LA	3RB	4LA	1RH	3LA	493,600,387	15,828
1RB	3LA	3LA	1RA	1RA	3LB	1LB	2LA	2RA	4RB	5LB	1RH	98,364,599	10,249
1RB	3LA	4LA	1RA	3RB	1RH	2LB	1LA	1LB	3RB	5RA	1RH	94,842,383	10,574

5 Behaviors of busy beavers

5.1 Introduction

How do good machines behave? We give below the tricks that allow them to reach high scores.

A *configuration* of the Turing machine M is a description of the tape. The position of the tape head and the state are indicated by writing together between parentheses the state and the symbol currently read by the tape head.

For example, the initial configuration on a blank tape is:

$$\ldots 0(A0)0 \ldots$$

We denote by a^k the string $a \ldots a$, k times. We write $C \vdash (t)\ D$ if the next move function leads from configuration C to configuration D in t computation steps.

5.2 Turing machines with 5 states and 2 symbols

5.2.1 Marxen and Buntrock's champion

This machine is the record holder in the Busy Beaver Competition for machines with 5 states and 2 symbols, since 1990.

It was analyzed by Buro (1990) (p. 64-67), and independently by Michel (1993).

Marxen and Buntrock (1990)
$s(M) = 47,176,870 =?\ S(5,2)$
$\sigma(M) = 4098 =?\ \Sigma(5,2)$

	0	1
A	1RB	1LC
B	1RC	1RB
C	1RD	0LE
D	1LA	1LD
E	1RH	0LA

Let $C(n) = \ldots 0(A0)1^n 0 \ldots$.

Then we have, for all $k \geq 0$,

$$\begin{array}{lcc} C(3k) & \vdash (5k^2+19k+15) & C(5k+6) \\ C(3k+1) & \vdash (5k^2+25k+27) & C(5k+9) \\ C(3k+2) & \vdash (6k+12) & \ldots 01(H0)1(001)^{k+1}10 \ldots \end{array}$$

So we have:

$$
\begin{array}{rl}
\multicolumn{2}{c}{\ldots 0(A0)0\ldots =} \\
C(0) & \vdash (15) \\
C(6) & \vdash (73) \\
C(16) & \vdash (277) \\
C(34) & \vdash (907) \\
C(64) & \vdash (2,757) \\
C(114) & \vdash (7,957) \\
C(196) & \vdash (22,777) \\
C(334) & \vdash (64,407) \\
C(564) & \vdash (180,307) \\
C(946) & \vdash (504,027) \\
C(1,584) & \vdash (1,403,967) \\
C(2,646) & \vdash (3,906,393) \\
C(4,416) & \vdash (10,861,903) \\
C(7,366) & \vdash (30,196,527) \\
C(12,284) & \vdash (24,576) \\
\multicolumn{2}{c}{\ldots 01(H0)1(001)^{4095}10\ldots}
\end{array}
$$

5.2.2 Marxen and Buntrock's runner-up

Marxen and Buntrock (1990)
$s(M) = 23,554,764$
$\sigma(M) = 4097$

	0	1
A	1RB	0LD
B	1LC	1RD
C	1LA	1LC
D	1RH	1RE
E	1RA	0RB

Let $C(n) = \ldots 0(A0)1^n0\ldots$.
Then we have, for all $k \geq 0$,

$$
\begin{array}{ccc}
C(3k) & \vdash (10k^2 + 10k + 4) & C(5k+3) \\
C(3k+1) & \vdash (3k+3) & \ldots 01(110)^k 11(H0)0\ldots \\
C(3k+2) & \vdash (10k^2 + 26k + 12) & C(5k+7)
\end{array}
$$

So we have:

$$
\begin{aligned}
&\dots 0(A0)0\dots = \\
C(0) &\vdash (4) \\
C(3) &\vdash (24) \\
C(8) &\vdash (104) \\
C(17) &\vdash (392) \\
C(32) &\vdash (1,272) \\
C(57) &\vdash (3,804) \\
C(98) &\vdash (11,084) \\
C(167) &\vdash (31,692) \\
C(282) &\vdash (89,304) \\
C(473) &\vdash (250,584) \\
C(792) &\vdash (699,604) \\
C(1,323) &\vdash (1,949,224) \\
C(2,208) &\vdash (5,424,324) \\
C(3,683) &\vdash (15,087,204) \\
C(6,142) &\vdash (6,144) \\
&\dots 01(110)^{2047}11(H0)0\dots
\end{aligned}
$$

5.3 Turing machines with 6 states and 2 symbols

5.3.1 Kropitz's machine found in June 2010

This machine is the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols, since June 2010.

Kropitz (2010)
$s(M)$ and $S(6,2) > 7.4 \times 10^{36534}$
$\sigma(M)$ and $\Sigma(6,2) > 3.5 \times 10^{18267}$

	0	1
A	1RB	1LE
B	1RC	1RF
C	1LD	0RB
D	1RE	0LC
E	1LA	0RD
F	1RH	1RC

Let $C(n) = \dots 0(A0)1^n0\dots$.
Then we have, for all $k \geq 0$,

$\dots 0(A0)0\dots$	$\vdash (29)$	$C(9)$
$C(3k+1)$	$\vdash (3k+3)$	$\dots 0111(011)^k(H0)0\dots$
$C(9k+9)$	$\vdash ((125 \times 16^{k+2} + 325 \times 4^{k+2} + 228k - 2289)/27)$	$C((50 \times 4^{k+1} - 11)/3)$
$C(9k+12)$	$\vdash ((125 \times 16^{k+2} + 325 \times 4^{k+2} + 228k - 912)/27)$	$C((50 \times 4^{k+1} + 1)/3)$

So we have:

$$
\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (29) \\
C(9) & \vdash (1293) \\
C(63) & \vdash (19,884,896,677) \\
C(273063) & \vdash (125 \times 16^{30341} + 325 \times 4^{30341} + 6,916,380)/27) \\
C(50 \times 4^{30340} + 1)/3) & \vdash (50 \times 4^{30340} + 7)/3) \\
 & \ldots 0111(011)^p(H0)0\ldots
\end{array}
$$

with $p = (50 \times 4^{30340} - 2)/9$.

So the total time is $s(M) = (125 \times 16^{30341} + 1750 \times 4^{30340} + 15)/27 + 19,885,154,163$, and the final number of 1 is $\sigma(M) = (25 \times 4^{30341} + 23)/9$.

Some configurations take a long time to halt. For example, $C(2) \vdash (t)$ END with $t > 10^{10^{10^{10^{18,705,352}}}}$. A proof of this fact is given by "Cloudy176" in
`http://googology.wikia.com/wiki/User_blog:Cloudy176/Proving_the_bound_for_S(7)`

This property was used by "Wythagoras", in March 2014, to define a (7,2)-TM that extends the present (6,2)-TM and enters this configuration C(2) in two steps. See
`http://googology.wikia.com/wiki/User_blog:Wythagoras/A_good_bound_for_S(7)%3F`

See detailed analysis in Michel (2015), Section 6.

5.3.2 Kropitz's machine found in May 2010

This machine was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols, from May 2010 to June 2010.

Kropitz (2010)
$s(M) > 3.8 \times 10^{21132}$
$\sigma(M) > 3.1 \times 10^{10566}$

	0	1
A	1RB	0LD
B	1RC	0RF
C	1LC	1LA
D	0LE	1RH
E	1LA	0RB
F	0RC	0RE

Analysis adapted from Shawn Ligocki:
Let $C(n,k) = \ldots 010^n1(C1)1^{3k}0\ldots$.
Then we have, for all $k \geq 0$, all $n \geq 0$,

$$
\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (47) & C(5,2) \\
C(0,k) & \vdash (3) & \ldots 01(H0)1^{3k+1}0\ldots \\
C(1,k) & \vdash (3k+37) & C(3k+2,2) \\
C(2,k) & \vdash (12k+44) & C(4,k+2) \\
C(3,k) & \vdash (3k+57) & C(3k+8,2) \\
C(n+4,k) & \vdash (27k^2+105k+112) & C(n,3k+5)
\end{array}
$$

So we have (the final configuration is reached in 22158 transitions):

$$\begin{array}{rl} \ldots 0(A0)0\ldots & \vdash (47) \\ C(5,2) & \vdash (430) \\ C(1,11) & \vdash (70) \\ C(35,2) & \vdash (430) \\ C(31,11) & \vdash (4,534) \\ C(27,38) & \vdash (43,090) \\ C(23,119) & \vdash (394,954) \\ C(19,362) & \vdash (3,576,310) \\ C(15,1091) & \vdash (32,252,254) \\ C(11,3278) & \vdash (290,466,970) \\ C(7,9839) & \vdash (2,614,793,074) \\ C(3,29522) & \vdash (88,623) \\ C(88574,2) & \vdash (430) \\ C(88570,11) & \vdash (4,534) \\ C(88566,38) & \vdash (43,090) \\ & \ldots \end{array}$$

Note that $C(4n+r,2) \vdash (t_n)\ C(r,u_n)$, with $u_n = (3^{n+2}-5)/2$, and $t_n = (3\times 9^{n+3} - 80 \times 3^{n+3} + 584n - 27)/32$.

Some configurations take a long time to halt. For example, $C(1,9) \vdash (t)$ END with $t > 10^{10^{10^{10^{10^{3520}}}}}$.

See detailed analysis in Michel (2015), Section 7.

5.3.3 Ligockis' machine found in December 2007

This machine was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols, from December 2007 to May 2010.

Terry and Shawn Ligocki (2007)
$s(M) > 2.5 \times 10^{2879}$
$\sigma(M) > 4.6 \times 10^{1439}$

	0	1
A	1RB	0LE
B	1LC	0RA
C	1LD	0RC
D	1LE	0LF
E	1LA	1LC
F	1LE	1RH

Let $C(n,p) = \ldots 0(A0)(10)^n R(\text{bin}(p))0\ldots$, where $R(\text{bin}(p))$ is the number p written in binary in reverse order, so that $C(n,4m+1) = C(n+1,m)$. The number of transitions between configurations $C(n,p)$ is infinite, but only 18 transitions are used in the computation on a blank tape. For all $m \geq 0$, all $k \geq 0$,

$$\begin{array}{ccc}
C(k,4m+3) & \vdash (4k+6) & C(k+2,m) \\
C(2k+1,4m) & \vdash (6k^2+52k+98) & C(3k+8,m) \\
C(4k,4m) & \vdash (24k^2+36k+13) & C(6k+2,2m+1) \\
C(4k+2,4m) & \vdash (24k^2+60k+27) & C(6k+2,128m+86) \\
C(k,8m+2) & \vdash (4k+14) & C(k+2,2m+1) \\
C(2k+1,32m+22) & \vdash (6k^2+64k+160) & C(3k+10,2m+1) \\
C(4k,32m+22) & \vdash (24k^2+36k+29) & C(6k+4,m) \\
C(4k+2,32m+22) & \vdash (24k^2+60k+43) & C(6k+2,1024m+342) \\
C(k,64m+46) & \vdash (4k+30) & C(k+4,m) \\
C(k+1,128m+6) & \vdash (8k+66) & C(k+6,2m+1) \\
C(2k,256m+14) & \vdash (6k^2+64k+172) & C(3k+11,m) \\
C(4k+1,256m+14) & \vdash (24k^2+84k+89) & C(6k+8,2m+1) \\
C(4k+3,256m+14) & \vdash (24k^2+108k+127) & C(6k+8,128m+86) \\
C(4k,512m+30) & \vdash (24k^2+156k+173) & C(6k+11,m) \\
C(4k+2,512m+30) & \vdash (24k^2+60k+57) & C(6k+2,16384m+11134) \\
C(4k+2,131072m+11134) & \vdash (24k^2+60k+89) & C(6k+2,4194304m+2848638) \\
C(4k,131072m+96126) & \vdash (24k^2+36k+109) & C(6k+10,m) \\
C(k+1,512m+94) & \vdash (2k+61) & ...0(10)^k1(H0)1110110101R(\text{bin}(m))0...
\end{array}$$

So we have (the final configuration is reached in 11026 transitions):

$$\begin{array}{rl}
& \ldots 0(A0)0\ldots = \\
C(0,0) & \vdash (13) \\
C(3,0) & \vdash (156) \\
C(11,0) & \vdash (508) \\
C(23,0) & \vdash (1396) \\
C(41,0) & \vdash (3538) \\
C(68,0) & \vdash (7,561) \\
C(105,0) & \vdash (19,026) \\
C(164,0) & \vdash (41,833) \\
C(249,0) & \vdash (98,802) \\
C(380,0) & \vdash (220,033) \\
C(573,0) & \vdash (505,746) \\
C(866,0) & \vdash (1,132,731) \\
C(1298,86) & \vdash (2,538,907) \\
C(1946,2390) & \vdash (5,697,907) \\
C(2918,76118) & \vdash (12,798,367) \\
C(4376,2435414) & \vdash (1,034,066,333) \\
C(6568,76106) & \vdash (26,286) \\
C(6570,19027) & \vdash (26,286) \\
C(6572,4756) & \vdash (64,845,937) \\
C(9860,2379) & \vdash (39,446) \\
C(9862,594) & \vdash (39,462) \\
C(9867,2) & \vdash (39,482) \\
& \ldots
\end{array}$$

5.3.4 Ligockis' machine found in November 2007

This machine was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols, from November to December 2007.

Terry and Shawn Ligocki (2007)
$s(M) > 8.9 \times 10^{1762}$
$\sigma(M) > 2.5 \times 10^{881}$

	0	1
A	1RB	0RF
B	0LB	1LC
C	1LD	0RC
D	1LE	1RH
E	1LF	0LD
F	1RA	0LE

Let $C(n,p) = \dots 0(F0)(10)^n R(\text{bin}(p))0\dots$, where $R(\text{bin}(p))$ is the number p written in binary in reverse order, so that $C(n, 4m+1) = C(n+1, m)$. The number of transitions between configurations $C(n,p)$ is infinite, but only 12 transitions are used in the computation on a blank tape. For all $m \geq 0$, all $k \geq 0$,

$$
\begin{array}{ccc}
\dots 0(A0)0\dots & \vdash (6) & C(0,15) \\
C(k,4m+3) & \vdash (4k+6) & C(k+2,m) \\
C(2k,4m) & \vdash (30k^2+20k+15) & C(5k+2,2m+1) \\
C(2k+1,4m) & \vdash (30k^2+40k+25) & C(5k+2,32m+20) \\
C(k,8m+2) & \vdash (8k+20) & C(k+3,2m+1) \\
C(2k,16m+6) & \vdash (30k^2+40k+23) & C(5k+2,32m+20) \\
C(2k+1,16m+6) & \vdash (30k^2+80k+63) & C(5k+7,2m+1) \\
C(k,32m+14) & \vdash (4k+18) & C(k+3,2m+1) \\
C(2k,128m+94) & \vdash (30k^2+40k+39) & C(5k+2,256m+84) \\
C(2k+1,128m+94) & \vdash (30k^2+80k+79) & C(5k+9,m) \\
C(k,256m+190) & \vdash (4k+34) & C(k+5,m) \\
C(k,512m+30) & \vdash (2k+43) & \dots 0(10)^k 1(H0)10100101R(\text{bin}(m))0\dots
\end{array}
$$

So we have (the final configuration is reached in 3346 transitions):

$$
\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (6) \\
C(0,15) & \vdash (6) \\
C(2,3) & \vdash (14) \\
C(4,0) & \vdash (175) \\
C(13,0) & \vdash (1,345) \\
C(32,20) & \vdash (8,015) \\
C(82,11) & \vdash (334) \\
C(84,2) & \vdash (692) \\
C(88,0) & \vdash (58,975) \\
C(223,0) & \vdash (374,095) \\
C(557,20) & \vdash (2,329,665) \\
C(1392,180) & \vdash (14,546,415) \\
C(3482,91) & \vdash (13,934) \\
C(3484,22) & \vdash (91,106,623) \\
C(8712,52) & \vdash (569,329,215) \\
C(21782,27) & \vdash (87,134) \\
C(21784,6) & \vdash (3,559,505,623) \\
C(54462,20) & \vdash (22,246,365,465) \\
C(136157,11) & \vdash (544,634) \\
C(136159,2) & \vdash (1,089,292) \\
C(136163,0) & \vdash (139,053,400,095) \\
C(340407,20) & \vdash (869,078,644,415) \\
 & \ldots
\end{array}
$$

See detailed analysis in Michel (2015), Section 8.

5.3.5 Marxen and Buntrock's machine found in March 2001

This machine was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols, from March 2001 to November 2007.

Marxen and Buntrock (2001)
$s(M) > 3.0 \times 10^{1730}$
$\sigma(M) > 1.2 \times 10^{865}$

	0	1
A	1RB	0LF
B	0RC	0RD
C	1LD	1RE
D	0LE	0LD
E	0RA	1RC
F	1LA	1RH

Let $C(n,p) = \ldots 0(A0)(01)^n R(\text{bin}(p))0\ldots$, where $R(\text{bin}(p))$ is the number p written in binary in reverse order, so that $C(n, 4m+2) = C(n+1, m)$. The number of transitions between configurations $C(n,p)$ is infinite, but only 20 transitions are used in the computation on a blank tape. For all $m \geq 0$, all $k \geq 0$,

$$
\begin{array}{ccc}
C(2k, 4m) & \vdash (9k^2+25k+9) & C(3k+1, 2m+1) \\
C(2k, 16m+1) & \vdash (9k^2+25k+17) & C(3k+2, 2m+1) \\
C(2k, 4m+3) & \vdash (9k^2+25k+9) & C(3k+1, 2m) \\
C(2k, 64m+53) & \vdash (9k^2+25k+25) & C(3k+3, 2m) \\
C(2k, 256m+9) & \vdash (9k^2+25k+29) & C(3k+4, 2m+1) \\
C(2k, 1024m+57) & \vdash (9k^2+25k+33) & C(3k+2, 128m+104) \\
C(2k, 1024m+85) & \vdash (9k^2+25k+41) & C(3k+5, 2m+1) \\
C(2k+1, 16m) & \vdash (9k^2+25k+21) & C(3k+3, 2m+1) \\
C(2k+1, 4m+1) & \vdash (9k^2+25k+13) & C(3k+1, 8m+4) \\
C(2k+1, 64m+4) & \vdash (9k^2+25k+29) & C(3k+4, 2m+1) \\
C(2k+1, 64m+3) & \vdash (9k^2+25k+25) & C(3k+1, 128m+104) \\
C(2k+1, 1024m+104) & \vdash (9k^2+43k+75) & C(3k+7, 2m+1) \\
C(2k+1, 16m+12) & \vdash (9k^2+25k+21) & C(3k+3, 2m) \\
C(2k+1, 16m+7) & \vdash (9k^2+25k+17) & C(3k+1, 32m+16) \\
C(2k+1, 256m+15) & \vdash (9k^2+25k+29) & C(3k+1, 512m+416) \\
C(2k+1, 64m+52) & \vdash (9k^2+25k+29) & C(3k+4, 2m) \\
C(2k+1, 256m+20) & \vdash (9k^2+25k+37) & C(3k+5, 2m+1) \\
C(2k+1, 4096m+420) & \vdash (9k^2+43k+89) & C(3k+8, 2m+1) \\
C(2k+1, 256m+211) & \vdash (9k^2+25k+33) & C(3k+1, 512m+168) \\
C(2k+1, 16m+11) & \vdash (9k^2+13k+10) & \ldots 0(10)^{3k+1}11(H0)10R(\mathrm{bin}(m))0\ldots
\end{array}
$$

So we have (the final configuration is reached in 4911 transitions):

$$
\begin{aligned}
&\ldots 0(A0)0\ldots = \\
C(0,0) &\vdash (9) \\
C(1,1) &\vdash (13) \\
C(1,4) &\vdash (29) \\
C(4,1) &\vdash (103) \\
C(8,1) &\vdash (261) \\
C(14,1) &\vdash (633) \\
C(23,1) &\vdash (1,377) \\
C(34,4) &\vdash (3,035) \\
C(52,3) &\vdash (6,743) \\
C(79,0) &\vdash (14,685) \\
C(120,1) &\vdash (33,917) \\
C(182,1) &\vdash (76,821) \\
C(275,1) &\vdash (172,359) \\
C(412,4) &\vdash (387,083) \\
C(619,3) &\vdash (867,079) \\
C(928,104) &\vdash (1,949,273) \\
C(1393,53) &\vdash (4,377,157) \\
C(2089,108) &\vdash (9,835,545) \\
C(3135,12) &\vdash (22,138,597) \\
C(4704,0) &\vdash (49,845,945) \\
C(7057,1) &\vdash (112,109,269) \\
C(10585,4) &\vdash (252,179,705) \\
&\ldots
\end{aligned}
$$

Note: Clive Tooth posted an analysis of this machine on Google Groups (sci.math>The Turing machine known as #r), on June 28, 2002. He used the configurations $S(n,x) = \ldots 0101(B1)010(01)^n x0\ldots$ His analysis can be easily connected to the present one, by noting that

$$C(n,p) \vdash (15)\ S(n-2, R(\text{bin}(p))).$$

5.3.6 Marxen and Buntrock's second machine

This machine was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols, from October 2000 to March 2001.

Marxen and Buntrock (2000)
$s(M) > 6.1 \times 10^{925}$
$\sigma(M) > 6.4 \times 10^{462}$

	0	1
A	1RB	0LB
B	0RC	1LB
C	1RD	0LA
D	1LE	1LF
E	1LA	0LD
F	1RH	1LE

Let $C(n) = \dots 01^n(B0)0\dots$.

Then we have, for all $k \geq 0$,

$\dots 0(A0)0\dots$	$\vdash (1)$	$C(1)$
$C(3k)$	$\vdash (54 \times 4^{k+1} - 27 \times 2^{k+3} + 26k + 86)$	$C(9 \times 2^{k+1} - 8)$
$C(3k+1)$	$\vdash (2048 \times (4^k - 1)/3 - 3 \times 2^{k+7} + 26k + 792)$	$C(2^{k+5} - 8)$
$C(3k+2)$	$\vdash (3k+8)$	$\dots 01(H1)(011)^k(0101)0\dots$

So we have:

$$\begin{array}{rl} \dots 0(A0)0\dots & \vdash (1) \\ C(1) & \vdash (408) \\ C(24) & \vdash (14,100,774) \\ C(4600) & \vdash (2048 \times (4^{1533} - 1)/3 - 3 \times 2^{1540} + 40650) \\ C(2^{1538} - 8) & \vdash (2^{1538} - 2) \\ & \dots 01(H1)(011)^p(0101)0\dots \end{array}$$

with $p = (2^{1538} - 10)/3$.

So the total time is $T = 2048 \times (4^{1533} - 1)/3 - 11 \times 2^{1538} + 14141831$, and the final number of 1 is $2 \times (2^{1538} - 10)/3 + 4$.

Note that

$$C(6k+1) \vdash (\ )\ C(3m) \vdash (\ )\ C(6p+4) \vdash (\ )\ C(3q+2) \vdash (\ )\ \text{END},$$

with $m = (2^{2k+5} - 8)/3$, $p = 3 \times 2^m - 2$, $q = (2^{2p+6} - 10)/3$.

So all configurations $C(n)$ lead to a halting configuration. Those taking the most time are $C(6k+1)$. For example:

$$C(7) \vdash (t)\ \text{END} \quad \text{with} \quad t > 10^{3.9 \times 10^{12}}.$$

More generally:

$$C(6k+1) \vdash (t(k))\ \text{END} \quad \text{with} \quad t(k) > 10^{10^{10^{(3k+2)/5}}}.$$

See also the analyses by Robert Munafo: the short one in
`http://mrob.com/pub/math/ln-notes1-4.html#mb6q`
and the detailed one in
`http://mrob.com/pub/math/ln-mb6q.html`
See detailed analysis in Michel (2015), Section 9.

5.3.7 Marxen and Buntrock's third machine

Marxen and Buntrock (2000)
$s(M) > 6.1 \times 10^{119}$
$\sigma(M) > 1.4 \times 10^{60}$

	0	1
A	1RB	0LC
B	1LA	1RC
C	1RA	0LD
D	1LE	1LC
E	1RF	1RH
F	1RA	1RE

Let $C(n,x) = \ldots 0(E0)1000(10)^n x0\ldots$, so that $C(n,10y) = C(n+1,y)$. The number of transitions between configurations $C(n,x)$ is infinite, but only 9 transitions are used in the computation on a blank tape. For all $k \geq 0$,

$$\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (18) & C(1,01) \\
C(2k,01^n) & \vdash (6k^2+22k+15) & C(3k+1,01^{n+1}) \\
C(2k,11) & \vdash (6k^2+34k+41) & C(3k+4,01) \\
C(2k,111) & \vdash (6k^2+34k+45) & C(3k+5,01) \\
C(2k,1111) & \vdash (6k^2+28k+25) & \ldots 01^{6k+11}(H0)0\ldots \\
C(2k+1,0) & \vdash (6k^2+34k+43) & C(3k+4,0) \\
C(2k+1,01) & \vdash (6k^2+22k+27) & C(3k+4,01) \\
C(2k+1,01^{n+2}) & \vdash (6k^2+22k+23) & C(3k+4,1^n) \\
C(2k+1,1^{n+2}) & \vdash (6k^2+34k+41) & C(3k+4,01^n)
\end{array}$$

So we have (the final configuration is reached in 337 transitions):

$$\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (18) \\
C(1,01) & \vdash (27) \\
C(4,01) & \vdash (83) \\
C(7,011) & \vdash (143) \\
C(13,0) & \vdash (463) \\
C(22,0) & \vdash (983) \\
C(34,01) & \vdash (2,123) \\
C(52,011) & \vdash (4,643) \\
C(79,0111) & \vdash (10,007) \\
C(122,0) & \vdash (23,683) \\
C(184,01) & \vdash (52,823) \\
C(277,011) & \vdash (117,323) \\
C(418,0) & \vdash (266,699) \\
C(628,01) & \vdash (598,499) \\
C(943,011) & \vdash (1,341,431) \\
C(1417,0) & \vdash (3,031,699) \\
C(2128,0) & \vdash (6,815,999) \\
C(3193,01) & \vdash (15,318,435) \\
C(4792,01) & \vdash (34,497,623) \\
C(7189,011) & \vdash (77,580,107) \\
C(10786,0) & \vdash (174,625,355) \\
 & \ldots
\end{array}$$

Note that, if $C(n,m) = \ldots 0(E0)1000(10)^n R(\text{bin}(m))0\ldots$, where $R(\text{bin}(m))$ is the number m written in binary in reverse order, so that $C(n,4m+1) = C(n+1,m)$, then we have also, for all k, $m \geq 0$,

$\ldots 0(A0)0 \ldots$	$\vdash (18)$	$C(1,2)$
$C(2k,2m)$	$\vdash (6k^2+22k+15)$	$C(3k+1,4m+2)$
$C(2k,32m+3)$	$\vdash (6k^2+34k+41)$	$C(3k+4,4m+2)$
$C(2k,128m+7)$	$\vdash (6k^2+34k+45)$	$C(3k+5,4m+2)$
$C(2k,32m+15)$	$\vdash (6k^2+28k+25)$	$\ldots 01^{6k+11}(H0)R(\text{bin}(m))0 \ldots$
$C(2k+1,4m)$	$\vdash (6k^2+34k+43)$	$C(3k+4,2m)$
$C(2k+1,32m+2)$	$\vdash (6k^2+22k+27)$	$C(3k+4,4m+2)$
$C(2k+1,8m+6)$	$\vdash (6k^2+22k+23)$	$C(3k+4,m)$
$C(2k+1,4m+3)$	$\vdash (6k^2+34k+41)$	$C(3k+4,2m)$

5.3.8 Another Marxen and Buntrock's machine

This machine was discovered in January 1990, and was published on the web (Google groups) on September 3, 1997. It was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols up to July 2000.

Marxen and Buntrock (1997)
$s(M) = 8,690,333,381,690,951$
$\sigma(M) = 95,524,079$

	0	1
A	1RB	1RA
B	1LC	1LB
C	0RF	1LD
D	1RA	0LE
E	1RH	1LF
F	0LA	0LC

Note the likeness to the machine N with 3 states and 3 symbols discovered, in August 2006, by Terry and Shawn Ligocki, and studied in Section 5.4.2. For this machine N, we have $s(N) = 4,345,166,620,336,565$ and $\sigma(N) = 95,524,079$, that is, same value of σ, and almost half the value of s. See analysis of this similarity in Section 5.9.

Analysis by Robert Munafo:
Let $C(n) = \ldots 0(D0)1^n0 \ldots$.
Then we have, for all $k \geq 0$,

$\ldots 0(A0)0 \ldots$	$\vdash (3)$	$C(2)$
$C(4k)$	$\vdash (8k+6)$	$\ldots 01(H0)(10)^{2k}110 \ldots$
$C(4k+1)$	$\vdash (20k^2+56k+30)$	$C(10k+9)$
$C(4k+2)$	$\vdash (20k^2+56k+33)$	$C(10k+9)$
$C(4k+3)$	$\vdash (20k^2+68k+51)$	$C(10k+12)$

So we have:

$$
\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (3) \\
C(2) & \vdash (33) \\
C(9) & \vdash (222) \\
C(29) & \vdash (1,402) \\
C(79) & \vdash (8,563) \\
C(202) & \vdash (52,833) \\
C(509) & \vdash (329,722) \\
C(1,279) & \vdash (2,056,963) \\
C(3,202) & \vdash (12,844,833) \\
C(8,009) & \vdash (80,272,222) \\
C(20,029) & \vdash (501,681,402) \\
C(50,079) & \vdash (3,135,358,563) \\
C(125,202) & \vdash (19,595,552,833) \\
C(313,009) & \vdash (122,471,892,222) \\
C(782,529) & \vdash (765,448,543,902) \\
C(1,956,329) & \vdash (4,784,051,443,102) \\
C(4,890,829) & \vdash (29,900,316,628,602) \\
C(12,227,079) & \vdash (186,876,942,247,563) \\
C(30,567,702) & \vdash (1,167,980,782,060,333) \\
C(76,419,259) & \vdash (7,299,879,658,619,323) \\
C(191,048,152) & \vdash (382,096,310) \\
 & \ldots 01(H0)(10)^{95524076}110\ldots
\end{array}
$$

5.4 Turing machines with 3 states and 3 symbols

5.4.1 Ligockis' champion

This machine is the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, since November 2007.

Terry and Shawn Ligocki (2007)
$s(M) = 119,112,334,170,342,540 =?\ S(3,3)$
$\sigma(M) = 374,676,383 =?\ \Sigma(3,3)$

	0	1	2
A	1RB	2LA	1LC
B	0LA	2RB	1LB
C	1RH	1RA	1RC

Let $C(n) = \ldots 0(A0)2^n 0\ldots$.
Then we have, for all $k \geq 0$,

$$
\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (3) & C(1) \\
C(8k+1) & \vdash (112k^2+116k+13) & C(14k+3) \\
C(8k+2) & \vdash (112k^2+144k+38) & C(14k+7) \\
C(8k+3) & \vdash (112k^2+172k+54) & C(14k+8) \\
C(8k+4) & \vdash (112k^2+200k+74) & C(14k+9) \\
C(8k+5) & \vdash (112k^2+228k+97) & \ldots 01(H1)2^{14k+9}0\ldots \\
C(8k+6) & \vdash (112k^2+256k+139) & C(14k+14) \\
C(8k+7) & \vdash (112k^2+284k+169) & C(14k+15) \\
C(8k+8) & \vdash (112k^2+312k+203) & C(14k+16)
\end{array}
$$

So we have (in 34 transitions):

$$\begin{array}{rl} \ldots 0(A0)0\ldots & \vdash (3) \\ C(1) & \vdash (13) \\ C(3) & \vdash (54) \\ C(8) & \vdash (203) \\ C(16) & \vdash (627) \\ C(30) & \vdash (1915) \\ & \quad \ldots \\ C(122,343,306) & \vdash (26,193,799,261,043,238) \\ C(214,100,789) & \vdash (80,218,511,093,348,089) \\ & \ldots 01(H1)2^{374676381}0\ldots \end{array}$$

See detailed analysis in Michel (2015), Section 3.

5.4.2 Ligockis' machine found in August 2006

This machine was the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, from August 2006 to November 2007.

Terry and Shawn Ligocki (2006)
$s(M) = 4,345,166,620,336,565$
$\sigma(M) = 95,524,079$

	0	1	2
A	1RB	2RC	1LA
B	2LA	1RB	1RH
C	2RB	2RA	1LC

Note the likeness to the machine N with 6 states and 2 symbols discovered, in January 1990, by Heiner Marxen and Jürgen Buntrock, and studied in Section 5.3.8. For this machine N, we have $s(N) = 8,690,333,381,690,951$ and $\sigma(N) = 95,524,079$, that is, same value of σ, and almost twice the value of s. See analysis of this similarity in Section 5.9.

Analysis by Shawn Ligocki:
Let $C(n,0) = \ldots 0(A0)12^n0\ldots$,
and $C(n,1) = \ldots 0(C0)12^n0\ldots$.
Then we have, for all $k \geq 0$,

$$\begin{array}{rll} C(2k,0) & \vdash (40k^2+32k+5) & C(5k+1,1) \\ C(2k+1,0) & \vdash (40k^2+82k+42) & \ldots 01^{10k+9}(H0)0\ldots \\ C(2k+1,1) & \vdash (40k^2+52k+19) & C(5k+3,1) \\ C(2k+2,1) & \vdash (40k^2+92k+53) & C(5k+5,0) \end{array}$$

So we have:

$$
\begin{array}{rl}
& \ldots 0(A0)0\ldots = \\
C(0,0) & \vdash (5) \\
C(1,1) & \vdash (19) \\
C(3,1) & \vdash (111) \\
C(8,1) & \vdash (689) \\
C(20,0) & \vdash (4,325) \\
C(51,1) & \vdash (26,319) \\
C(128,1) & \vdash (164,609) \\
C(320,0) & \vdash (1,029,125) \\
C(801,1) & \vdash (6,420,819) \\
C(2003,1) & \vdash (40,132,111) \\
C(5008,1) & \vdash (250,830,689) \\
C(12520,0) & \vdash (1,567,704,325) \\
C(31301,1) & \vdash (9,797,713,819) \\
C(78253,1) & \vdash (61,235,789,611) \\
C(195633,1) & \vdash (382,723,880,691) \\
C(489083,1) & \vdash (2,392,024,743,391) \\
C(1222708,1) & \vdash (14,950,155,868,889) \\
C(3056770,0) & \vdash (93,438,477,237,325) \\
C(7641926,1) & \vdash (582,990,375,746,317) \\
C(19104815,0) & \vdash (3,649,939,963,043,376) \\
& \ldots 01^{95524079}(H0)0\ldots
\end{array}
$$

5.4.3 Lafitte and Papazian's machine found in April 2006

This machine was the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, from April to August 2006.

Lafitte and Papazian (2006)
$s(M) = 4,144,465,135,614$
$\sigma(M) = 2,950,149$

	0	1	2
A	1RB	1RH	2LC
B	1LC	2RB	1LB
C	1LA	2RC	2LA

Let $C(n,0) = \ldots 0(A0)1^n0\ldots$,
and $C(n,1) = \ldots 0(A0)1^n210\ldots$.

Then we have, for all $k \geq 0$ (note the likeness to Brady's machine of Section 5.4.8),

$$
\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (16) & C(6,0) \\
C(2k+1,0) & \vdash (4k+5) & \ldots 01(H2)2^{2k}10\ldots \\
C(2k+2,0) & \vdash (10k^2+27k+23) & C(5k+6,1) \\
C(2k,1) & \vdash (10k^2+27k+18) & C(5k+5,1) \\
C(2k+1,1) & \vdash (10k^2+51k+60) & C(5k+12,0)
\end{array}
$$

So we have:

$$
\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (16) \\
C(6,0) & \vdash (117) \\
C(16,1) & \vdash (874) \\
C(45,1) & \vdash (6,022) \\
C(122,0) & \vdash (37,643) \\
C(306,1) & \vdash (238,239) \\
C(770,1) & \vdash (1,492,663) \\
C(1930,1) & \vdash (9,338,323) \\
C(4830,1) & \vdash (58,387,473) \\
C(12080,1) & \vdash (364,979,098) \\
C(30205,1) & \vdash (2,281,474,302) \\
C(75522,0) & \vdash (14,259,195,543) \\
C(188806,1) & \vdash (89,121,812,989) \\
C(472020,1) & \vdash (557,013,573,288) \\
C(1180055,1) & \vdash (3,481,348,698,727) \\
C(2950147,0) & \vdash (5,900,297) \\
\multicolumn{2}{c}{\ldots 01(H2)2^{2950146}10\ldots}
\end{array}
$$

Note that we have also, for all $k \geq 0$,

$$
\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (133) & C(16,1) \\
C(2k,1) & \vdash (10k^2+27k+18) & C(5k+5,1) \\
C(4k+1,1) & \vdash (290k^2+737k+468) & C(25k+31,1) \\
C(4k+3,1) & \vdash (40k^2+162k+158) & \ldots 01(H2)2^{10k+16}10\ldots
\end{array}
$$

5.4.4 Lafitte and Papazian's machine found in September 2005

This machine was the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, from September 2005 to April 2006.

Lafitte and Papazian (2005)
$s(M) = 987,522,842,126$
$\sigma(M) = 1,525,688$

	0	1	2
A	1RB	2LA	1RA
B	1RC	2RB	0RC
C	1LA	1RH	1LA

Let $C(n,0) = \ldots 0(A0)2^n 0\ldots$,
and $C(n,1) = \ldots 0(A0)2^n 10\ldots$.
Then we have, for all $k \geq 0$,

$$
\begin{array}{ccc}
C(4k,0) & \vdash (14k^2+16k+5) & C(7k+2,1) \\
C(4k+1,0) & \vdash (14k^2+30k+15) & C(7k+5,0) \\
C(4k+2,0) & \vdash (14k^2+30k+15) & C(7k+5,0) \\
C(4k+3,0) & \vdash (14k^2+44k+35) & C(7k+9,1) \\
C(2k+1,1) & \vdash (4k+3) & \ldots 01(12)^k 01(H0)0\ldots \\
C(4k,1) & \vdash (14k^2+26k+11) & C(7k+4,0) \\
C(4k+2,1) & \vdash (14k^2+40k+29) & C(7k+8,1)
\end{array}
$$

So we have:

$$
\begin{aligned}
& \ldots 0(A0)0\ldots = \\
C(0,0) & \vdash (5) \\
C(2,1) & \vdash (29) \\
C(8,1) & \vdash (119) \\
C(18,0) & \vdash (359) \\
C(33,0) & \vdash (1,151) \\
C(61,0) & \vdash (3,615) \\
C(110,0) & \vdash (11,031) \\
C(194,0) & \vdash (33,711) \\
C(341,0) & \vdash (103,715) \\
C(600,0) & \vdash (317,405) \\
C(1052,1) & \vdash (975,215) \\
C(1845,0) & \vdash (2,989,139) \\
C(3232,0) & \vdash (9,153,029) \\
C(5658,1) & \vdash (28,048,133) \\
C(9906,1) & \vdash (85,927,133) \\
C(17340,1) & \vdash (263,203,871) \\
C(30349,0) & \vdash (806,103,591) \\
C(53114,0) & \vdash (2,468,672,331) \\
C(92951,0) & \vdash (7,560,436,829) \\
C(162668,1) & \vdash (23,154,325,799) \\
C(284673,0) & \vdash (70,910,514,191) \\
C(498181,0) & \vdash (217,164,134,715) \\
C(871820,0) & \vdash (665,064,835,635) \\
C(1525687,1) & \vdash (3,051,375) \\
& \ldots 01(12)^{762843}01(H0)0\ldots
\end{aligned}
$$

5.4.5 Lafitte and Papazian's machine found in August 2005

This machine was the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, from August to September 2005.

Lafitte and Papazian (2005)
$s(M) = 4,939,345,068$
$\sigma(M) = 107,900$

	0	1	2
A	1RB	1RH	2RB
B	1LC	0LB	1RA
C	1RA	2LC	1RC

Let $C(n,0) = \ldots 0(C0)2^n0\ldots$,
and $C(n,1) = \ldots 0(C0)2^n10\ldots$.

Then we have, for all $k \geq 0$,

$$\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (3) & C(1,1) \\
C(4k,0) & \vdash (14k^2+16k+5) & C(7k+2,1) \\
C(4k+1,0) & \vdash (14k^2+22k+7) & C(7k+3,0) \\
C(4k+2,0) & \vdash (14k^2+30k+15) & C(7k+5,0) \\
C(4k+3,0) & \vdash (14k^2+36k+23) & C(7k+7,1) \\
C(2k,1) & \vdash (2k+2) & \ldots 01(21)^k 1(H0)0\ldots \\
C(4k+1,1) & \vdash (14k^2+20k+9) & C(7k+3,1) \\
C(4k+3,1) & \vdash (14k^2+34k+21) & C(7k+6,0)
\end{array}$$

So we have:

$$\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (3) \\
C(1,1) & \vdash (9) \\
C(3,1) & \vdash (21) \\
C(6,0) & \vdash (59) \\
C(12,0) & \vdash (179) \\
C(23,1) & \vdash (541) \\
C(41,0) & \vdash (1,627) \\
C(73,0) & \vdash (4,939) \\
C(129,0) & \vdash (15,047) \\
C(227,0) & \vdash (45,943) \\
C(399,1) & \vdash (140,601) \\
C(699,0) & \vdash (430,151) \\
C(1225,1) & \vdash (1,317,033) \\
C(2145,1) & \vdash (4,032,873) \\
C(3755,1) & \vdash (12,349,729) \\
C(6572,0) & \vdash (37,818,579) \\
C(11503,1) & \vdash (115,816,521) \\
C(20131,0) & \vdash (354,675,511) \\
C(35231,1) & \vdash (1,086,184,945) \\
C(61655,0) & \vdash (3,326,402,857) \\
C(107898,1) & \vdash (107,900) \\
\multicolumn{2}{c}{\ldots 01(21)^{53949}1(H0)0\ldots}
\end{array}$$

5.4.6 Souris's machine for $S(3,3)$

This machine was the record holder in the Busy Beaver Competition for $S(3,3)$, from July to August 2005.

Souris (2005)
$s(M) = 544,884,219$
$\sigma(M) = 32,213$

	0	1	2
A	1RB	1LB	2LA
B	1LA	1RC	1RH
C	0LA	2RC	1LC

Let $C(n,0) = \ldots 0(A0)1^n 0\ldots$,
and $C(n,1) = \ldots 0(A0)1^n 20\ldots$.

Then we have, for all $k \geq 0$,

$$\begin{array}{lcc}
\ldots 0(A0)0 \ldots & \vdash (4) & C(3,0) \\
C(3k+2,0) & \vdash (21k^2+43k+19) & \ldots 011(H2)2^{7k+1}0 \ldots \\
C(3k+3,0) & \vdash (21k^2+43k+24) & C(7k+7,0) \\
C(3k+4,0) & \vdash (21k^2+43k+26) & C(7k+7,1) \\
C(3k+1,1) & \vdash (21k^2+61k+35) & \ldots 011(H2)2^{7k+3}0 \ldots \\
C(3k+2,1) & \vdash (21k^2+61k+42) & C(7k+9,0) \\
C(3k+3,1) & \vdash (21k^2+61k+46) & C(7k+9,1)
\end{array}$$

So we have:

$$\begin{array}{rl}
\ldots 0(A0)0 \ldots & \vdash (4) \\
C(3,0) & \vdash (24) \\
C(7,0) & \vdash (90) \\
C(14,1) & \vdash (622) \\
C(37,0) & \vdash (3,040) \\
C(84,1) & \vdash (17,002) \\
C(198,1) & \vdash (92,736) \\
C(464,1) & \vdash (507,472) \\
C(1087,0) & \vdash (2,752,290) \\
C(2534,1) & \vdash (15,010,582) \\
C(5917,0) & \vdash (81,666,440) \\
C(13804,1) & \vdash (444,833,917) \\
\ldots 011(H2)2^{32210}0 \ldots &
\end{array}$$

5.4.7 Souris's machine for $\Sigma(3,3)$

This machine was the record holder in the Busy Beaver Competition for $\Sigma(3,3)$, from July to August 2005.

Souris (2005)
$s(M) = 310,341,163$
$\sigma(M) = 36,089$

	0	1	2
A	1RB	2RA	2RC
B	1LC	1RH	1LA
C	1RA	2LB	1LC

Let $C(n,0) = \ldots 0(C0)1^n0\ldots$,
and $C(n,1) = \ldots 0(C0)1^n210\ldots$.
Then we have, for all $k \geq 0$,

$$\begin{array}{lcc}
\ldots 0(A0)0 \ldots & \vdash (4) & C(1,1) \\
C(2k+2,0) & \vdash (5k^2+32k+17) & C(5k+5,0) \\
C(2k+3,0) & \vdash (5k^2+32k+21) & C(5k+4,1) \\
C(2k+1,1) & \vdash (5k^2+32k+15) & C(5k+4,0) \\
C(2k+2,1) & \vdash (5k^2+37k+30) & \ldots 012^{5k+5}1(H2)10 \ldots
\end{array}$$

So we have:

$$
\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (4) \\
C(1,1) & \vdash (15) \\
C(4,0) & \vdash (54) \\
C(10,0) & \vdash (225) \\
C(25,0) & \vdash (978) \\
C(59,1) & \vdash (5,148) \\
C(149,0) & \vdash (29,002) \\
C(369,1) & \vdash (175,183) \\
C(924,0) & \vdash (1,077,374) \\
C(2310,0) & \vdash (6,695,525) \\
C(5775,0) & \vdash (41,737,353) \\
C(14434,1) & \vdash (260,620,302) \\
\multicolumn{2}{c}{\ldots 012^{36085}1(H2)10\ldots}
\end{array}
$$

Note that we have also, for all $k \geq 0$,

$$
\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (19) & C(4,0) \\
C(2k+2,0) & \vdash (5k^2+32k+17) & C(5k+5,0) \\
C(4k+3,0) & \vdash (145k^2+299k+93) & \ldots 012^{25k+10}1(H2)10\ldots \\
C(4k+5,0) & \vdash (145k^2+444k+281) & C(25k+24,0)
\end{array}
$$

5.4.8 Brady's machine

This machine was the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, from December 2004 to July 2005.

Brady (2004)
$s(M) = 92,649,163$
$\sigma(M) = 13,949$

	0	1	2
A	1RB	1RH	2LC
B	1LC	2RB	1LB
C	1LA	0RB	2LA

Let $C(n,0) = \ldots 0(A0)1^n0\ldots$,
and $C(n,1) = \ldots 0(A0)1^n210\ldots$.
Then we have, for all $k \geq 0$,

$$
\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (6) & C(0,1) \\
C(2k+1,0) & \vdash (4k+5) & \ldots 01(H2)2^{2k}10\ldots \\
C(2k+2,0) & \vdash (10k^2+15k+10) & C(5k+3,1) \\
C(2k,1) & \vdash (10k^2+27k+18) & C(5k+5,1) \\
C(2k+1,1) & \vdash (10k^2+51k+60) & C(5k+12,0)
\end{array}
$$

So we have:

$$\begin{array}{rl} \ldots 0(A0)0\ldots & \vdash (6) \\ C(0,1) & \vdash (18) \\ C(5,1) & \vdash (202) \\ C(22,0) & \vdash (1,160) \\ C(53,1) & \vdash (8,146) \\ C(142,0) & \vdash (50,060) \\ C(353,1) & \vdash (318,796) \\ C(892,0) & \vdash (1,986,935) \\ C(2228,1) & \vdash (12,440,056) \\ C(5575,1) & \vdash (77,815,887) \\ C(13947,0) & \vdash (27,897) \\ \multicolumn{2}{c}{\ldots 01(H2)2^{13946}10\ldots} \end{array}$$

Note that we have also, for all $k \geq 0$,

$$\begin{array}{ccc} \ldots 0(A0)0\ldots & \vdash (6) & C(0,1) \\ C(2k,1) & \vdash (10k^2+27k+18) & C(5k+5,1) \\ C(4k+1,1) & \vdash (290k^2+677k+395) & C(25k+28,1) \\ C(4k+3,1) & \vdash (40k^2+162k+158) & \ldots 01(H2)2^{10k+16}10\ldots \end{array}$$

5.5 Turing machines with 2 states and 4 symbols

5.5.1 Ligockis' champion

This machine is the record holder in the Busy Beaver Competition for machines with 2 states and 4 symbols, since February 2005.

Terry and Shawn Ligocki (2005)
$s(M) = 3,932,964 =?\ S(2,4)$
$\sigma(M) = 2,050 =?\ \Sigma(2,4)$

	0	1	2	3
A	1RB	2LA	1RA	1RA
B	1LB	1LA	3RB	1RH

Let $C(n,1) = \ldots 0(A0)2^n10\ldots$,
and $C(n,2) = \ldots 0(A0)2^n110\ldots$.
Then we have, for all $k \geq 0$,

$$\begin{array}{ccc} \ldots 0(A0)0\ldots & \vdash (6) & C(1,2) \\ C(3k,1) & \vdash (15k^2+9k+3) & C(5k+1,1) \\ C(3k+1,1) & \vdash (15k^2+24k+13) & \ldots 013^{5k+2}1(H1)0\ldots \\ C(3k+2,1) & \vdash (15k^2+29k+17) & C(5k+4,2) \\ C(3k,2) & \vdash (15k^2+11k+3) & C(5k+1,2) \\ C(3k+1,2) & \vdash (15k^2+21k+7) & C(5k+3,1) \\ C(3k+2,2) & \vdash (15k^2+36k+23) & \ldots 013^{5k+4}1(H1)0\ldots \end{array}$$

So we have:

$$
\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (6) \\
C(1,2) & \vdash (7) \\
C(3,1) & \vdash (27) \\
C(6,1) & \vdash (81) \\
C(11,1) & \vdash (239) \\
C(19,2) & \vdash (673) \\
C(33,1) & \vdash (1,917) \\
C(56,1) & \vdash (5,399) \\
C(94,2) & \vdash (15,073) \\
C(158,1) & \vdash (42,085) \\
C(264,2) & \vdash (117,131) \\
C(441,2) & \vdash (325,755) \\
C(736,2) & \vdash (905,527) \\
C(1228,1) & \vdash (2,519,044) \\
\ldots 013^{2047}1(H1)0\ldots &
\end{array}
$$

See detailed analysis in Michel (2015), Section 4.

5.5.2 Brady's runner-up

This machine was the record holder in the Busy Beaver Competition for machines with 2 states and 4 symbols, from 1988 to February 2005.

Brady (1988)		0	1	2	3
$s(M) = 7,195$	A	1RB	3LA	1LA	1RA
$\sigma(M) = 90$	B	2LA	1RH	3RA	3RB

Let $C(n,0) = \ldots 0(A0)3^n0\ldots$,
and $C(n,1) = \ldots 0(A0)3^n20\ldots$.

Then we have, for all $k \geq 0$,

$$
\begin{array}{rlc}
C(3k,0) & \vdash (15k^2+7k+3) & C(5k+1,1) \\
C(3k+1,0) & \vdash (15k^2+22k+11) & \ldots 013^{5k+1}1(H0)0\ldots \\
C(3k+2,0) & \vdash (15k^2+27k+13) & C(5k+4,0) \\
C(3k,1) & \vdash (15k^2+28k+16) & \ldots 013^{5k+3}1(H0)0\ldots \\
C(3k+1,1) & \vdash (15k^2+33k+19) & C(5k+5,0) \\
C(3k+2,1) & \vdash (15k^2+43k+33) & C(5k+7,1)
\end{array}
$$

So we have:

$$
\begin{array}{rl}
\ldots 0(A0)0\ldots = & \\
C(0,0) & \vdash (3) \\
C(1,1) & \vdash (19) \\
C(5,0) & \vdash (55) \\
C(9,0) & \vdash (159) \\
C(16,1) & \vdash (559) \\
C(30,0) & \vdash (1,573) \\
C(51,1) & \vdash (4,827) \\
\ldots 013^{88}1(H0)0\ldots &
\end{array}
$$

5.6 Turing machines with 2 states and 5 symbols

5.6.1 Ligockis' champion

This machine is the record holder in the Busy Beaver Competition for machines with 2 states and 5 symbols, since November 2007.

Terry and Shawn Ligocki (2007)
$s(M)$ and $S(2,5) > 1.9 \times 10^{704}$
$\sigma(M)$ and $\Sigma(2,5) > 1.7 \times 10^{352}$

	0	1	2	3	4
A	1RB	2LA	1RA	2LB	2LA
B	0LA	2RB	3RB	4RA	1RH

Let $C(n,1) = \ldots 013^n(B0)0\ldots$,
and $C(n,2) = \ldots 023^n(B0)0\ldots$,
and $C(n,3) = \ldots 03^n(B0)0\ldots$,
and $C(n,4) = \ldots 04113^n(B0)0\ldots$,
and $C(n,5) = \ldots 04123^n(B0)0\ldots$,
and $C(n,6) = \ldots 0413^n(B0)0\ldots$,
and $C(n,7) = \ldots 0423^n(B0)0\ldots$,
and $C(n,8) = \ldots 043^n(B0)0\ldots$.
Then we have, for all $k \geq 0$,

$\ldots 0(A0)0\ldots$	$\vdash (1)$	$C(0,1)$
$C(2k,1)$	$\vdash (3k^2+8k+4)$	$C(3k+1,1)$
$C(2k+1,1)$	$\vdash (3k^2+8k+4)$	$C(3k+1,2)$
$C(2k,2)$	$\vdash (3k^2+14k+9)$	$C(3k+2,1)$
$C(2k+1,2)$	$\vdash (3k^2+8k+4)$	$C(3k+2,3)$
$C(2k,3)$	$\vdash (3k^2+8k+2)$	$C(3k,1)$
$C(2k+1,3)$	$\vdash (3k^2+8k+22)$	$C(3k+1,4)$
$C(2k,4)$	$\vdash (3k^2+8k+8)$	$C(3k+3,1)$
$C(2k+1,4)$	$\vdash (3k^2+8k+4)$	$C(3k+1,5)$
$C(2k,5)$	$\vdash (3k^2+14k+13)$	$C(3k+4,1)$
$C(2k+1,5)$	$\vdash (3k^2+8k+4)$	$C(3k+2,6)$
$C(2k,6)$	$\vdash (3k^2+8k+6)$	$C(3k+2,1)$
$C(2k+1,6)$	$\vdash (3k^2+8k+4)$	$C(3k+1,7)$
$C(2k,7)$	$\vdash (3k^2+14k+11)$	$C(3k+3,1)$
$C(2k+1,7)$	$\vdash (3k^2+8k+4)$	$C(3k+2,8)$
$C(2k,8)$	$\vdash (3k^2+8k+4)$	$C(3k+1,1)$
$C(2k+1,8)$	$\vdash (3k^2+5k+3)$	$\ldots 01(H2)2^{3k}0\ldots$

So we have:

$$
\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (1) \\
C(0,1) & \vdash (4) \\
C(1,1) & \vdash (4) \\
C(1,2) & \vdash (4) \\
C(2,3) & \vdash (13) \\
C(3,1) & \vdash (15) \\
C(4,2) & \vdash (49) \\
C(8,1) & \vdash (84) \\
C(13,1) & \vdash (160) \\
C(19,2) & \vdash (319) \\
C(29,3) & \vdash (722) \\
C(43,4) & \vdash (1495) \\
C(64,5) & \vdash (3533) \\
C(100,1) & \vdash (7904) \\
\ldots &
\end{array}
$$

See detailed analysis in Michel (2015), Section 5.

5.6.2 Ligockis' machine found in August 2006

This machine was the record holder in the Busy Beaver Competition for machines with 2 states and 5 symbols, from August 2006 to October 2007.

Terry and Shawn Ligocki (2006)
$s(M) = 7,069,449,877,176,007,352,687$
$\sigma(M) = 172,312,766,455$

	0	1	2	3	4
A	1RB	0RB	4RA	2LB	2LA
B	2LA	1LB	3RB	4RA	1RH

Analysis by Shawn Ligocki:
Let $C(n,1) = \ldots 03^n(B0)0\ldots$,
and $C(n,2) = \ldots 013^n(B0)0\ldots$,
and $C(n,3) = \ldots 01403^n(B0)0\ldots$,
and $C(n,4) = \ldots 01413^n(B0)0\ldots$.
Then we have, for all $k \geq 0$,

$$
\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (1) & C(0,2) \\
C(2k,1) & \vdash (5k^2+14k+3) & C(5k+1,2) \\
C(2k+1,1) & \vdash (5k^2+14k+7) & C(5k+3,2) \\
C(2k,2) & \vdash (5k^2+14k+3) & C(5k+1,1) \\
C(2k+1,2) & \vdash (5k^2+14k+11) & C(5k+2,3) \\
C(2k,3) & \vdash (5k^2+14k+3) & C(5k+1,4) \\
C(2k+1,3) & \vdash (5k^2+14k+9) & C(5k+4,1) \\
C(2k,4) & \vdash (5k^2+14k+3) & C(5k+1,3) \\
C(2k+1,4) & \vdash (5k^2+9k+4) & \ldots 011(H1)2^{5k+2}0\ldots
\end{array}
$$

So we have (in 30 transitions):

$$\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (1) \\
C(0,2) & \vdash (3) \\
C(1,1) & \vdash (7) \\
C(3,2) & \vdash (30) \\
C(7,3) & \vdash (96) \\
C(19,1) & \vdash (538) \\
 & \ldots \\
C(4411206821,1) & \vdash (24,323,432,041,896,588,247) \\
C(11028017053,2) & \vdash (152,021,450,201,199,582,755) \\
C(27570042632,3) & \vdash (950,134,063,605,862,157,707) \\
C(68925106581,4) & \vdash (5,938,337,896,640,612,100,114) \\
 & \ldots 011(H1)2^{172312766452}0\ldots
\end{array}$$

5.6.3 Lafitte and Papazian's machine found in June 2006

This machine was the record holder in the Busy Beaver Competition for $\Sigma(2,5)$, from June to August 2006.

G. Lafitte and C. Papazian (2006)
$s(M) = 14,103,258,269,249$
$\sigma(M) = 4,848,239$

	0	1	2	3	4
A	1RB	3LB	4LB	4LA	2RA
B	2LA	1RH	3RB	4RA	3RB

Let $C(n,1) = \ldots 0132^n33(B0)0\ldots$,
and $C(n,2) = \ldots 01342^n33(B0)0\ldots$,
and $C(n,3) = \ldots 0142^n33(B0)0\ldots$,
and $C(n,4) = \ldots 012^n33(B0)0\ldots$.

Then we have, for all $k \geq 0$,

$$\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (10) & C(0,1) \\
C(2k,1) & \vdash (3k^2+12k+15) & C(3k+2,2) \\
C(2k+1,1) & \vdash (3k^2+12k+11) & C(3k+2,3) \\
C(2k,2) & \vdash (3k^2+12k+9) & C(3k+2,1) \\
C(2k+1,2) & \vdash (3k^2+18k+30) & C(3k+5,2) \\
C(2k,3) & \vdash (3k^2+12k+9) & C(3k+2,4) \\
C(2k+1,3) & \vdash (3k^2+18k+28) & C(3k+4,2) \\
C(2k,4) & \vdash (3k^2+12k+13) & C(3k+1,2) \\
C(2k+1,4) & \vdash (3k^2+9k+5) & \ldots 01(H4)4^{3k+2}20\ldots
\end{array}$$

So we have (in 36 transitions):

$$\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (10) \\
C(0,1) & \vdash (15) \\
C(2,2) & \vdash (24) \\
C(5,1) & \vdash (47) \\
C(8,3) & \vdash (105) \\
C(14,4) & \vdash (244) \\
 & \ldots \\
C(957674,2) & \vdash (687,860,363,760) \\
C(1436513,1) & \vdash (1,547,683,663,691) \\
C(2154770,3) & \vdash (3,482,288,243,304) \\
C(3232157,4) & \vdash (7,835,138,850,959) \\
\ldots 01(H4)4^{4848236}20\ldots &
\end{array}$$

5.6.4 Lafitte and Papazian's machine found in May 2006

This machine was the record holder in the Busy Beaver Competition for machines with 2 states and 5 symbols, from May to June 2006.

G. Lafitte and C. Papazian (2006)
$s(M) = 3,793,261,759,791$
$\sigma(M) = 2,576,467$

	0	1	2	3	4
A	1RB	3RA	4LB	2RA	3LA
B	2LA	1RH	4RB	4RB	2LB

Let $C(n,1) = \ldots 014^n(B0)0\ldots$,
and $C(n,2) = \ldots 034^n(B0)0\ldots$.
Then we have, for all $k \geq 0$,

$$\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (1) & C(0,1) \\
C(3k,1) & \vdash (4k^2+17k+11) & C(4k+3,1) \\
C(3k+1,1) & \vdash (4k^2+25k+20) & C(4k+4,1) \\
C(3k+2,1) & \vdash (4k^2+17k+13) & C(4k+3,2) \\
C(3k,2) & \vdash (4k^2+17k+11) & C(4k+3,1) \\
C(3k+1,2) & \vdash (4k^2+25k+20) & C(4k+4,1) \\
C(3k+2,2) & \vdash (4k^2+21k+24) & \ldots 01(H2)23^{4k+3}20\ldots
\end{array}$$

So we have (in 45 transitions):

$$\begin{array}{rl}
\dots 0(A0)0\dots & \vdash (1) \\
C(0,1) & \vdash (11) \\
C(3,1) & \vdash (32) \\
C(7,1) & \vdash (86) \\
C(12,1) & \vdash (143) \\
C(19,1) & \vdash (314) \\
 & \dots \\
C(815207,1) & \vdash (295,364,260,408) \\
C(1086943,2) & \vdash (525,094,796,254) \\
C(1449260,1) & \vdash (933,496,546,059) \\
C(1932347,2) & \vdash (1,659,550,059,339) \\
\multicolumn{2}{c}{\dots 01(H2)23^{2576463}20\dots}
\end{array}$$

Note that we have also, for all $k \geq 0$,

$$\begin{array}{ccc}
\dots 0(A0)0\dots & \vdash (1) & C(0,1) \\
C(3k,1) & \vdash (4k^2+17k+11) & C(4k+3,1) \\
C(3k+1,1) & \vdash (4k^2+25k+20) & C(4k+4,1) \\
C(9k+2,1) & \vdash (100k^2+151k+45) & C(16k+7,1) \\
C(9k+5,1) & \vdash (100k^2+239k+120) & C(16k+12,1) \\
C(9k+8,1) & \vdash (100k^2+279k+186) & \dots 01(H2)23^{16k+15}20\dots
\end{array}$$

Note: The machine obtained by replacing B4 → 2LB by B4 → 3LB has the same behavior but final configuration $\dots 01(H3)3^{2576464}20\dots$.

5.6.5 Lafitte and Papazian's machine found in December 2005

This machine was the record holder in the Busy Beaver Competition for $S(2,5)$, from December 2005 to May 2006.

G. Lafitte and C. Papazian (2005)
$s(M) = 924,180,005,181$
$\sigma(M) = 1,137,477$

	0	1	2	3	4
A	1RB	3RA	1LA	1LB	3LB
B	2LA	4LB	3RA	2RB	1RH

Let $C(n,1) = \dots 012^n(B0)0\dots$,
and $C(n,2) = \dots 032^n(B0)0\dots$.

Then we have, for all $k \geq 0$,

$$\begin{array}{ccc}
\dots 0(A0)0\dots & \vdash (69) & C(8,1) \\
C(2k+1,1) & \vdash (15k^2+37k+31) & \dots 01221(H1)1^{5k+1}20\dots \\
C(2k+2,1) & \vdash (15k^2+32k+19) & C(5k+3,2) \\
C(2k,2) & \vdash (15k^2+32k+19) & C(5k+3,1) \\
C(2k+1,2) & \vdash (15k^2+62k+70) & C(5k+9,1)
\end{array}$$

So we have:

$$
\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (69) \\
C(8,1) & \vdash (250) \\
C(18,2) & \vdash (1,522) \\
C(48,1) & \vdash (8,690) \\
C(118,2) & \vdash (54,122) \\
C(298,1) & \vdash (333,315) \\
C(743,2) & \vdash (2,087,687) \\
C(1864,1) & \vdash (13,031,226) \\
C(4658,2) & \vdash (81,438,162) \\
C(11648,1) & \vdash (508,796,290) \\
C(29118,2) & \vdash (3,179,933,122) \\
C(72798,1) & \vdash (19,873,380,815) \\
C(181993,2) & \vdash (124,209,722,062) \\
C(454989,1) & \vdash (776,311,217,849) \\
\multicolumn{2}{c}{\ldots 01221(H1)1^{1137471}20\ldots}
\end{array}
$$

Note that we have also, for all $k \geq 0$,

$$
\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (69) & C(8,1) \\
C(2k+1,1) & \vdash (15k^2+37k+31) & \ldots 01221(H1)1^{5k+1}20\ldots \\
C(4k+2,1) & \vdash (435k^2+524k+166) & C(25k+14,1) \\
C(4k+4,1) & \vdash (435k^2+884k+453) & C(25k+23,1)
\end{array}
$$

5.6.6 Lafitte and Papazian's machine found in October 2005

This machine was the record holder in the Busy Beaver Competition for $\Sigma(2,5)$, from October 2005 to May 2006.

G. Lafitte and C. Papazian (2005)
$s(M) = 912,594,733,606$
$\sigma(M) = 1,957,771$

	0	1	2	3	4
A	1RB	3LB	1RH	1LA	1LA
B	2LA	3RB	4LB	4LB	3RA

Let $C(n,1) = \ldots 0(A0)1^n 20\ldots$,
and $C(n,2) = \ldots 0(A0)1^n 40\ldots$,
and $C(n,3) = \ldots 0(A0)1^n 320\ldots$.
Then we have, for all $k \geq 0$,

$$
\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (11) & C(3,1) \\
C(2k+1,1) & \vdash (5k^2+28k+26) & C(5k+6,1) \\
C(2k+2,1) & \vdash (5k^2+18k+11) & C(5k+3,2) \\
C(2k,2) & \vdash (5k^2+18k+11) & C(5k+3,1) \\
C(2k+1,2) & \vdash (5k^2+18k+13) & C(5k+3,3) \\
C(2k+1,3) & \vdash (5k^2+18k+9) & C(5k+3,1) \\
C(2k+2,3) & \vdash (5k^2+23k+17) & \ldots 013^{5k+4}1(H0)0\ldots
\end{array}
$$

So we have:

$$\begin{array}{rl}
\ldots 0(A0)0\ldots & \vdash (11) \\
C(3,1) & \vdash (59) \\
C(11,1) & \vdash (291) \\
C(31,1) & \vdash (1,571) \\
C(81,1) & \vdash (9,146) \\
C(206,1) & \vdash (53,867) \\
C(513,2) & \vdash (332,301) \\
C(1283,3) & \vdash (2,065,952) \\
C(3208,1) & \vdash (12,876,910) \\
C(8018,2) & \vdash (80,432,578) \\
C(20048,1) & \vdash (502,483,070) \\
C(50118,2) & \vdash (3,140,218,478) \\
C(125298,1) & \vdash (19,624,987,195) \\
C(313243,2) & \vdash (122,653,507,396) \\
C(783108,3) & \vdash (766,577,764,781) \\
\ldots 013^{1957769}1(H0)0\ldots &
\end{array}$$

Note that we have also, for all $k \geq 0$,

$$\begin{array}{ccc}
\ldots 0(A0)0\ldots & \vdash (11) & C(3,1) \\
C(2k+1,1) & \vdash (5k^2+28k+26) & C(5k+6,1) \\
C(2k+2,1) & \vdash (5k^2+18k+11) & C(5k+3,2) \\
C(2k,2) & \vdash (5k^2+18k+11) & C(5k+3,1) \\
C(4k+1,2) & \vdash (145k^2+176k+45) & C(25k+8,1) \\
C(4k+3,2) & \vdash (145k^2+321k+167) & \ldots 013^{25k+19}1(H0)0\ldots
\end{array}$$

5.7 Collatz-like problems

Sameness of behaviors of the Turing machines above is striking. Their behaviors depend on transitions in the following form:

$$C(ak+b) \vdash (\ )\ C(ck+d),$$

where a, c are fixed, and $b = 0, \ldots, a-1$. Sometimes, another parameter is added: $C(ak+b, p)$.

These transitions can be compared to the following problem. Let T be defined by

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (3x+1)/2 & \text{if } x \text{ is odd.} \end{cases}$$

This can also be written

$$\begin{array}{rcl} T(2m) & = & m \\ T(2m+1) & = & 3m+2 \end{array}$$

When T is iterated over positive integers, do we always reach the loop: $T(2) = 1$, $T(1) = 2$? This question is a famous open problem in mathematics, called $3x+1$ problem, or *Collatz problem*.

A similar question can be asked about iterating transitions of configurations $C(ak+b,p)$ on positive integers. Do the iterated transitions always reach a halting configuration? For all the machines above (except for the machine with 6 states and 2 symbols in Section 5.3.6), this question is presently an open problem in mathematics. Because of likeness to Collatz problem, these problems are called *Collatz-like problems*. Thus, for each machine above (except for the machine with 6 states and 2 symbols in Section 5.3.6), the halting problem (that is, on what inputs does this machine stop?) depends on an open Collatz-like problem.

5.8 Non-Collatz-like behaviors

Some Turing machines run a large number of steps on a small piece of tape. Such machines do not seem to be Collatz-like. We list below some interesting machines with this sort of behavior.

5.8.1 Turing machines with 3 states and 3 symbols

A. H. Brady (November 2004)
$s(M) = 2,315,619$
$\sigma(M) = 31$

	0	1	2
A	1RB	2LB	1LC
B	1LA	2RB	1RB
C	1RH	2LA	0LC

Brady called this machine "Surprise-in-a-Box".

See also the simulation by Heiner Marxen:
`http://www.drb.insel.de/~heiner/BB/simAB3Y_SB.html`

5.8.2 Turing machines with 2 states and 5 symbols

(a) First machine

G. Lafitte and C. Papazian (July 2006)
$s(M) = 26,375,397,569,930$
$\sigma(M) = 143$

	0	1	2	3	4
A	1RB	3LA	1LA	4LA	1RA
B	2LB	2RA	1RH	0RA	0RB

This machine was the record holder for $S(2,5)$, from July to August 2006.

See also the simulation by Heiner Marxen:
`http://www.drb.insel.de/~heiner/BB/simLaf25_j.html`

(b) Second machine

G. Lafitte and C. Papazian (July 2006)
$s(M) = 7,021,292,621$
$\sigma(M) = 37$

	0	1	2	3	4
A	1RB	4LA	1LA	1RH	2RB
B	2LB	3LA	1LB	2RA	0RB

5.9 Turing machines in distinct classes with similar behaviors

In this section, we give examples of machines that have similar behaviors, but not the same numbers of states and symbols.

5.9.1 (2,4)-TM and (3,3)-TM

Terry and Shawn Ligocki (2005)
$s(M) = 3,932,964 =?\ S(2,4)$
$\sigma(M) = 2,050 =?\ \Sigma(2,4)$

	0	1	2	3
A	1RB	2LA	1RA	1RA
B	1LB	1LA	3RB	1RH

This machine is the record holder in the Busy Beaver Competition for machines with 2 states and 4 symbols, since February 2005.

A. H. Brady (2004)
$s(M) = 3,932,964$
$\sigma(M) = 2,050$

	0	1	2
A	1RB	1LC	1RH
B	1LA	1LC	2RB
C	1RB	2LC	1RC

There is a step-by-step correspondence between the configurations of these machines.

5.9.2 (6,2)-TM and (3,3)-TM

Marxen and Buntrock (1997)
$s(M) = 8,690,333,381,690,951$
$\sigma(M) = 95,524,079$

	0	1
A	1RB	1RA
B	1LC	1LB
C	0RF	1LD
D	1RA	0LE
E	1RH	1LF
F	0LA	0LC

This machine was discovered in January 1990, and was published on the web (Google groups) on September 3, 1997. It was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols up to July 2000.

Terry and Shawn Ligocki (2006)
$s(M) = 4,345,166,620,336,565$
$\sigma(M) = 95,524,079$

	0	1	2
A	1RB	2RC	1LA
B	2LA	1RB	1RH
C	2RB	2RA	1LC

This machine was the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, from August 2006 to November 2007.

Note that these machines have same σ value, and the s value of the first one is almost twice the s value of the second one.

The behaviors of these machines can be related as follows.
Given the analyses of the (6,2)-TM in Section 5.3.8 and the (3,3)-TM in Section 5.4.2, the following functions f and g can be defined:

$$\begin{cases} f(4k) & \text{undefined}, \\ f(4k+1) = 10k+9, \\ f(4k+2) = 10k+9, \\ f(4k+3) = 10k+12. \end{cases} \qquad \begin{cases} g(2k,0) = (5k+1,1), \\ g(2k+1,0) & \text{undefined}, \\ g(2k,1) = (5k,0), \\ g(2k+1,1) = (5k+3,1). \end{cases}$$

Now, let h be defined by

$$h(n,0) = 10n+2,$$
$$h(n,1) = 10n-1.$$

Then: $h \circ g = f \circ h$.

There is no step-by-step correspondence between these machines, but there is a phase correspondence, according to functions f and g.

6 Properties of the busy beaver functions

6.1 Growth properties

- Rado (1962) defined $S(n)$ and $\Sigma(n)$, that are denoted in this article $S(n,2)$ and $\Sigma(n,2)$.
- These functions grow faster than any computable function. Formally, for any computable function f, there is an integer N such that, for any integer $n > N$,

  $$S(n) > \Sigma(n) > f(n)$$

  This was proved by Rado (1962) who defined these functions in order to get noncomputable functions.
- It is easy to prove that the two variables functions $S(n,k)$ and $\Sigma(n,k)$ are increasing with the number n of states if the number k of symbols is constant. Formally, for any integer $k \geq 2$, if $n > m$, then

  $$S(n,k) > S(m,k) \qquad \text{and} \qquad \Sigma(n,k) > \Sigma(m,k)$$
- As Harland (2016b) noticed, the same result for the number of symbols, with a constant number of states, is far from obvious, and still unproven. Petersen (2017) proved that functions $S(n,k)$ and $\Sigma(n,k)$ are increasing with the number k of symbols if the number n of states is sufficiently large. The proof uses *introspective encoding*, a tool developped by Ben-Amram and Petersen (2002).

6.2 Relations between the busy beaver functions

- Rado (1962) proved that

  $$S(n) < (n+1)\Sigma(5n)2^{\Sigma(5n)}.$$

- Julstrom (1993) proved that

$$S(n) < \Sigma(28n).$$

- Julstrom (1992) proved that

$$S(n) < \Sigma(20n).$$

- Wang and Xu (1995) proved that

$$S(n) < \Sigma(10n).$$

- In an unpublished technical report in German, Buro (1990) (p. 5-6) proved that

$$S(n) < \Sigma(9n).$$

- Yang, Ding and Xu (1997) proved that

$$S(n) < \Sigma(8n),$$

  and that there is a constant c such that

$$S(n) < \Sigma(3n + c).$$

- Ben-Amram, Julstrom and Zwick (1996) proved that

$$S(n) < \Sigma(3n + 6),$$

  and

$$S(n) < (2n - 1)\Sigma(3n + 3).$$

- Ben-Amram and Petersen (2002) proved that there is a constant c such that

$$S(n) < \Sigma(n + 8n/\log_2 n + c).$$

7 Variants of busy beavers

7.1 Busy beavers defined by 4-tuples

The Turing machines used for regular busy beavers are based on 5-tuples. For example, the initial transition is

(A,0) $\longrightarrow$ (1,R,B)

and generally a transition is

(state, scanned symbol) $\longrightarrow$ (new written symbol, move of the head, new state)

Instead of both writing a symbol and moving the head in one transition, these actions can be split up into two transitions, in the form of a 4-tuple:

(state, scanned symbol) $\longrightarrow$ (new written symbol **or** move of the head, new state)

This alternative definition was introduced by Post in 1947 (Recursive unsolvability of a problem of Thue, *The Journal of Symbolic Logic*, Vol. 12, 1-11). So Turing machines defined by 4-tuples are also called *Post machines*, or *Post-Turing machines*.

A busy beaver competition for such machines was studied by Oberschelp, Schmidt-Göttsch and Todt (1988), who defined two busy beaver functions, for the number of non-blank symbols, and for the number of steps, and gave some values and lower bounds for these functions.

The busy beaver competition for such machines are also studied by P. Machado and F. Pereira, see
`http://fmachado.dei.uc.pt/publications`
and B. van Heuveln and his team, see
`http://www.cogsci.rpi.edu/~heuveb/Research/BB/index.html`

In their book, Boolos and Jeffrey (1974) used the 4-tuples variant to display the busy beaver problem.

Harland (2016b) tackled 4-tuples machines in
`http://arxiv.org/abs/1610.03184`
He gave a proof of the following theorem:

Theorem. For any n-state, m-symbol, 4-tuples machine M, halting on a blank tape, there exists a n-state, m-symbol, 5-tuples machine N, halting on a blank tape, such that $\sigma(N) = \sigma(M)$, that is, with the same number of non-blank symbols written on the tape when it halts.

Moreover, the proof provides a simple algorithm that transforms a 4-tuples machine into an equivalent 5-tuples machine. So Harland concludes that searching for 5-tuples machines subsumes searching for 4-tuples machines.

7.2 Busy beavers whose head can stand still

In the definition of the Turing machines used for regular busy beavers, the tape head has to move one cell right or left at each step, and cannot stand still. If we allow the tape head to stand still, new machines come into the competition, and they can beat the current champions.

So Norbert Bátfai found, in August 2009, a Turing machine M with 5 states and 2 symbols with $s(M) = 70{,}740{,}810$ and $\sigma(M) = 4098$. See

`http://arxiv.org/abs/0908.4013`

This machine beats the current champion for the number of steps ($s = 47{,}176{,}870$). It seems that relaxing this condition on moves does not allow us to obtain machines with behaviors different from those of regular busy beavers. But the study is still to be done.

7.3 Busy beavers on a one-way infinite tape

In the definition of the Turing machines used for regular busy beavers, the tape is infinite on both left and right sides. Walsh (1982) considered Turing machines with one-way infinite tape. Initially, the tape head scans the first (leftmost) tape cell. A Turing machine halts either by entering a halting state or by falling off the left end of the tape, that is, moving left from cell 1. If a Turing machine M halts when it starts from a blank tape, its score is defined to be k if the rightmost tape cell ever visited by M's head is the kth cell from the left. $\Sigma(n, m)$ is defined as the largest score of all halting n-state, m-symbol Turing machines. Walsh proved that, with this definition, $\Sigma(2, 3) = 6$.

7.4 Two-dimensional busy beavers

The Turing machines used for regular busy beavers have a one-dimensional tape. Turing machines with two-dimensional or higher-dimensional tapes were first defined by Hartmanis and Stearns in 1965 (On the computational complexity of algorithms, *Transactions of the AMS*, Vol. 117, 285-306).

Brady (1988) launched the busy beaver competition for two-dimensional Turing machines. He also defined, first, "TurNing machines", where the head reorients itself at each step, and, second, machines that work on a triangular grid.

Tim Hutton resumed the search for two-dimensional busy beavers. See

`https://github.com/GollyGang/ruletablerepository/wiki/TwoDimensionalTuringMachines`

He gave the following results:

For $S_2(k, n)$: (k states, n symbols)

3 symbols	38	?		
2 symbols	6	32	4632 ?	25,772,988,638 ?
	2 states	3 states	4 states	5 states

For $\Sigma_2(k, n)$: (k states, n symbols)

3 symbols	10	?		
2 symbols	4	11	244 ?	935,508,401 ?
	2 states	3 states	4 states	5 states

Note that

$$S_2(3, 2) = 32 > S(3, 2) = 21,$$

and

$$\Sigma_2(3,2) = 11 > \Sigma(3,2) = 6.$$

Tim Hutton also studied higher-dimensional machines and found that, for all $n > 0$, $S_n(2,2) = 6$ and $\Sigma_n(2,2) = 4$.

He also studied one-dimensional and higher-dimensional Turing machines with *relative movements*, that is, where the head has an orientation and reorients itself at each step.

8 The methods

The machines presented in this paper were discovered by means of computer programs. These programs contain procedures that achieve the following tasks:

1. To enumerate Turing machines without repetition.
2. To simulate Turing machines efficiently.
3. To recognize non-halting Turing machines.

Note that these procedures are often mixed together in real programs as follows: A tree of transition tables is generated, and, as soon as some transitions are defined, the corresponding Turing machine is simulated. If the definition of a new transition is necessary, the tree is extended. If the computation seems to loop, a proof of this fact is provided.

If the purpose is to prove a value for the busy beaver functions, then all Turing machines in a class have to be studied. The machines that pass through the three procedures above are either halting machines, from which the better one is selected, or holdouts waiting for better programs or for hand analyses.

If the purpose is to find lower bounds, a systematic enumeration of machines is not necessary. Terry and Shawn Ligocki said they used simulated annealing to find some of their machines.

The following references can be consulted for more information:

- Brady (1983) and Machlin and Stout (1990) for (4,2)-TM,
- Marxen and Buntrock (1990) and Hertel (2009) for (5,2)-TM,
- Lafitte and Papazian (2007) for (2,3)-TM,
- Page about Macro Machines on Marxen's website. See `http://www.drb.insel.de/~heiner/BB/macro.html`
- Harland (2016a) and Harland (2016b).

9 Busy beavers and unprovability

9.1 The result

Let $S(n) = S(n, 2)$ be Rado's busy beaver function. We know that $S(2) = 6$, $S(3) = 21$, $S(4) = 107$, and we can hope to prove that $S(5) = 47,176,870$. As we will see below, the fact that the busy beaver function S is not computable implies that it is not possible to prove that, for any natural number n, $S(n)$ has its true value.

Formally, we have the following theorem.

Theorem. *Let T be a well-known mathematical theory such as Peano arithmetic (PA) or Zermelo-Fraenkel set theory with axiom of choice (ZFC). Then there exist numbers N and L such that $S(N) = L$, but the sentence "$S(N) = L$" is not provable in T.*

This theorem is an easy consequence of the following proposition.

Proposition. *Let T be a well-known mathematical theory such as PA or ZFC. Then there exists a Turing machine with two symbols M that does not stop when it is launched on a blank tape, but the fact that it does not stop is not provable in T.*

Proof of the theorem from the proposition. Let M be the Turing machine given by the proposition, let N be the number of states of M, and let $L = S(N)$. Then, to prove that "$S(N) = L$", we have to prove that M does not stop. But, by the proposition, such a proof does not exist.

Note that, if "$S(N) = L$" is a true sentence unprovable in theory T, then, for all $m > L$, "$S(N) < m$" is also a true sentence unprovable in theory T.

In the following, we consider many kinds of proofs of the proposition and of the theorem.

9.2 A direct proof

This proposition is well-known and a one line proof can be given, as follows.

Proof. If all non-halting machines were provably non-halting, then an algorithm that gives simultaneously the computable enumeration of the halting machines and the computable enumeration of the provably non-halting machines would solve the halting problem on a blank tape.

We give a detailed proof for nonspecialist readers.

Detailed proof. Let $M_1, M_2, \ldots$ be a computably enumerable sequence of all Turing machines with two symbols. Such a sequence can be obtained as follows: we list machines according to their number of states, and, inside the set of machines with n states, we list the machines according to the alphabetical order of their transition tables.

Let $T_1, T_2, \ldots$ be a computably enumerable sequence of the theorems of the theory T. The existence of such a sequence is the main requirement that theory T has to satisfy in order that the proposition holds, and of course such a sequence exists for well-known mathematical theories such as PA or ZFC.

Now consider the following algorithms A and B.

Algorithm A. We launch the machines M_i on the blank tape as follows:

- one step of computation of M_1,

- 2 steps of computation of M_1, 2 steps of computation of M_2,
- 3 steps of computation of M_1, 3 steps of computation of M_2, 3 steps of computation of M_3,
- ...

When a machine M_i stops, we add it to a list of machines that stop when they are launched on a blank tape.

Note that, given a machine M, by running Algorithm A we will know that M stops if M stops, but we will never know that M doesn't stop if M doesn't stop.

Algorithm B. We launch the algorithm that provides the computably enumerable sequence of theorems of theory T, and each time we get a theorem T_i, we look and see if this is a theorem of the form "The Turing machine M does not stop when it is launched on a blank tape". If that is the case, we add M to a list of Turing machines that provably do not stop on a blank tape.

Note that, given a machine M, by running Algorithm B we will know that M is provably non-halting if M is provably non-halting, but we will never know that M is not provably non-halting if M is not provably non-halting.

Now we have two algorithms, A and B, and

- Algorithm A gives us a computably enumerable list of the Turing machines that stop when they are launched on a blank tape.
- Algorithm B gives us a computably enumerable list of the Turing machines that provably do not stop on a blank tape.

We mix together these two algorithms, by a procedure called dovetailing, to get Algorithm C, as follows.

Algorithm C.

- one step of Algorithm A, one step of Algorithm B,
- 2 steps of Algorithm A, 2 steps of Algorithm B,
- 3 steps of Algorithm A, 3 steps of Algorithm B,
- ...

Algorithm C gives us simultaneously both the computably enumerable lists provided by Algorithm A and Algorithm B.

So Algorithm C gives us both the list of halting Turing machines and the list of provably non-halting Turing machines (on a blank tape).

Now we are ready to prove the proposition. If all non-halting Turing machines were provably non-halting, then Algorithm C would give us the list of halting Turing machines and the list of non-halting Turing machines (on a blank tape). So, given a Turing machine M, by running Algorithm C, we would see M appearing in one of the lists, and we could settle the halting problem for machine M on a blank tape. So Algorithm C would give us a computable procedure to settle the halting problem on a blank tape. But it is known that such a computable procedure does not exist. Thus, there exists a non-halting Turing machine that is not provably non-halting on a blank tape.

9.3 The proposition as a special case of a general result

The proposition is a special case of the following theorem.

Theorem. *Let A be a set of natural numbers that is computably enumerable but not computable, and let T be a well-known mathematical theory such as PA or ZFC. Then there exists a natural number n such that the sentence "n is not a member of A" is true but not provable in theory T.*

Proof. Since A is computably enumerable, there exists an algorithm that enumerates the natural numbers in A. If all natural numbers not in A were provably not in A, then, by enumerating the proofs of theorems of theory T, we would get an algorithm that enumerates the natural numbers not in A. By running simultaneously both these algorithms, we could get a procedure that decides membership in A, contradicting the fact that A is not computable.

The proposition is obtained from this theorem by numbering the list of Turing machines, and by defining A as the set of numbers of Turing machines that stop on a blank tape.

9.4 Some theoretical examples of Turing machines that satisfy the proposition

Consider the Turing machine M given by the proposition: M does not stop when it is launched on a blank tape, but this fact is not provable in theory T. Can we get an idea of what such a machine M looks like? We give below some examples of such a Turing machine.

9.4.1 Example 1: Using Gödel's Second Incompleteness Theorem

Let M be a machine that enumerates the theorems of theory T, and stops when it finds a contradiction (such as $0 = 1$ if T is Peano arithmetic).

Then a proof within theory T that M does not stop would be a proof within theory T of the consistency of T, which is impossible by Gödel's Second Incompleteness Theorem (if theory T is consistent).

9.4.2 Example 2: Using Gödel's First Incompleteness Theorem

Another example can be given using Gödel's First Incompleteness Theorem. If T is PA or ZFC, supposed to be consistent, the proof of this theorem provides a formula F that asserts its own unprovability. Thus F is true, but unprovable within theory T.

Consider the machine M that enumerates the theorems of theory T, and stops when it finds formula F. Machine M does not stop, since F is unprovable, but a proof that it does not stop would be a proof that F is unprovable, so, since F *is* "F is unprovable", a proof of F, which is impossible, since F is unprovable.

9.4.3 Example 3: Using the Recursion Theorem

As a third example, consider the machine M that enumerates the theorems of theory T (PA or ZFC, supposed to be consistent), and stops when it finds a formula F that says that M itself does not stop. Such a machine can be proved to exist by applying the Recursion

Theorem to the function f such that machine $M_{f(x)}$ stops if it finds a proof that machine M_x does not stop.

Then F is true, because, if F were false, then M would stop, so F would be a theorem of T, so F would be true. But F is unprovable, because since F is true, M does not stop, so F is not a theorem of theory T. So the fact that M does not stop is true and unprovable.

9.5 Some explicit examples of Turing machines that satisfy the proposition

Since May 2016, there are explicit constructions of Turing machines whose behaviors are independent of ZFC. These machines never halt on a blank tape, but this fact cannot be proved in ZFC.

9.5.1 Example 1: Yedidia and Aaronson's machine

Adam Yedidia and Scott Aaronson gave, in May 2016, a Turing machine with 7910 states and two symbols such as it cannot be proved in ZFC that it never halts. They note that enumerating the theorems of ZFC would need a big number of states. They use a graph theoretic statement that Harvey Friedman proved to be equivalent to the consistency of a theory that implies the consistency of ZFC. By using a new high-level language that is easily compiled down to Turing machine description, they build a machine that would halt if it finds a counterexample to Friedman's statement. See Yedidia and Aaronson (2016).

9.5.2 Example 2: O'Rear's machine

S. O'Rear improved the number of states to 1919, in September 2016. His machine enumerates the theorems of a formal system which has the same power as ZFC. See

`https://github.com/sorear/metamath-turing-machines`

For a general presentation, see also Scott Aaronson's blog, available at

`http://www.scottaaronson.com/blog/?p=2725`

9.6 A proof using Kolmogorov complexity

There is another proof of unprovability, based on Kolmogorov complexity. The Kolmogorov complexity of a number is the length of the shortest program from which a universal Turing machine can output this number. By Chaitin's Incompleteness Theorem, for any well-known mathematical theory T, there exists a number $n(T)$ such that, for all numbers of complexity greater than $n(T)$, the fact that they have complexity greater than $n(T)$ is true but unprovable within theory T.

Chaitin's theorem also applies to the complexity defined as follows: The complexity of a number k is the smallest number n of states of a Turing machine with n states and two symbols that outputs this number k, written as a string of k symbols 1, when the machine is launched on a blank tape.

So there exists a number $n(T)$ such that, for any number k of complexity greater than $n(T)$, the sentence "the complexity of k is greater than $n(T)$" is true but unprovable within theory T. But "$k > \Sigma(n(T))$" implies "the complexity of k is greater than $n(T)$", so, for any number $k > \Sigma(n(T))$, the sentence "$k > \Sigma(n(T))$" is true but unprovable within theory T.

For more details, see Chaitin (1987), Boolos, Burgess and Jeffrey (2002), p. 230, who note that $n(T) < 10 \uparrow\uparrow 10$, a stack of 10 powers of 10, and Lafitte (2009).

Acknowledgments

This historical survey would not have been possible without the contributions of many people. In particular, many thanks to Norbert Bátfai, Allen Brady, Daniel Briggs, Serge Grigorieff, James Harland, Tim Hutton, Grégory Lafitte, Shawn Ligocki, Terry Ligocki, Maurice Margenstern, Heiner Marxen, Armando Matos, Robert Munafo, Christophe Papazian, Holger Petersen, Myron Souris, H.J.M. Wijers, and Hector Zenil.

References

1. Ben-Amram A.M., Julstrom B.A. and Zwick U. (1996)
   A note on busy beavers and other creatures
   Mathematical Systems Theory **29** (4), July-August 1996, 375–386.

2. Ben-Amram A.M. and Petersen H. (2002)
   Improved bounds for functions related to busy beavers
   Theory of Computing Systems **35** (1), January-February 2002, 1–11.

3. Boolos G.S., Burgess J.P. and Jeffrey R.C. (2002)
   Computability and Logic, 4th Ed.
   Cambridge, 2002.

4. Boolos G.S. and Jeffrey R.C. (1974)
   Computability and Logic
   Cambridge, 1974.

5. Brady A.H. (1964)
   Solutions to restricted cases of the halting problem
   Ph.D. Thesis, Oregon State University, Corvallis, December 1964.

6. Brady A.H. (1965)
   Solutions of restricted cases of the halting problem used to determine particular values of a non-computable function (abstract)
   Notices of the AMS **12** (4), June 1965, 476–477.

7. Brady A.H. (1966)
   The conjectured highest scoring machines for Rado's $\Sigma(k)$ for the value $k = 4$
   IEEE Transactions on Electronic Computers, EC-**15**, October 1966, 802–803.

8. Brady A.H. (1974)
   UNSCC Technical Report 11-74-1, November 1974.

9. Brady A.H. (1975)
   The solution to Rado's busy beaver game is now decided for $k = 4$ (abstract)
   Notices of the AMS **22** (1), January 1975, A-25.

10. Brady A.H. (1983)
The determination of the value of Rado's noncomputable function $\Sigma(k)$ for four-state Turing machines
Mathematics of Computation **40** (162), April 1983, 647–665.

11. Brady A.H. (1988)
The busy beaver game and the meaning of life
in: The Universal Turing Machine: A Half-Century Survey, R. Herken (Ed.), Oxford University Press, 1988, 259–277.

12. Buro M. (1990)
Ein Beitrag zur Bestimmung von Rados $\Sigma(5)$ oder Wie fängt man fleißige Biber? (German)
[A contribution to the determination of Rado's $\Sigma(5)$, or: How to catch a busy beaver?]
Technical Report 146, Rheinisch-Westfälische Technische Hochschule, Aachen, November 1990
(available at `https://skatgame.net/mburo/ps/diploma.pdf`).

13. Chaitin G. (1987)
Computing the busy beaver function
in: Open Problems in Communication and Computation, Springer, 1987, 108-112.
Reprinted in: Information, Randomness and Incompleteness, World Scientifics, 2nd Ed., 1990
(available at `http://logic.amu.edu.pl/images/6/6c/Bobr.pdf`).

14. Dewdney A.K. (1984a)
Computer recreations
Scientific American **251** (2), August 1984, 10–17.

15. Dewdney A.K. (1984b)
Computer recreations
Scientific American **251** (5), November 1984, 27.

16. Dewdney A.K. (1985a)
Computer recreations
Scientific American **252** (3), March 1985, 19.

17. Dewdney A.K. (1985b)
Computer recreations
Scientific American **252** (4), April 1985, 16.

18. Green M.W. (1964)
A lower bound on Rado's sigma function for binary Turing machines
in: *Proceedings of the 5th IEEE Annual Symposium on Switching Circuit Theory and Logical Design*, November 1964, 91–94.

19. Harland J. (2013)
Busy beaver machines and the observant otter heuristic
in: *Proceedings of the 19th Computing: The Australasian Theory Symposium (CATS 2013)*, January 2013.

20. Harland J. (2016a)
Busy beaver machines and the observant otter heuristic (or how to tame dreadful dragons)
Theoretical Computer Science **646**, 20 September 2016, 61–85
(preprint available at `https://arxiv.org/abs/1602.03228`).

21. Harland J. (2016b)
Generating candidate busy beaver machines (or how to build the zany zoo)
(available at `https://arxiv.org/abs/1610.03184`).

22. Hertel J. (2009)
Computing the uncomputable Rado sigma function
The Mathematica Journal **11** (2), 2009, 270–283.

23. Jones J.P. (1974)
Recursive undecidability – an exposition
The American Mathematical Monthly **81** (7), September 1974, 724–738.

24. Julstrom B.A. (1992)
A bound on the shift function in terms of the busy beaver function
SIGACT News **23** (3), Summer 1992, 100–106.

25. Julstrom B.A. (1993)
Noncomputability and the busy beaver problem
The UMAP Journal **14** (1), Spring 1993, 39–74.

26. Kopp R.J. (1981)
The busy beaver problem
M.A. Thesis, State University of New York, Binghamton, 1981.

27. Korfhage R.R. (1966)
Logic and Algorithms: With applications to the computer and information sciences
Wiley, 1966.

28. Lafitte G. (2009)
Busy beavers gone wild
in: The Complexity of Simple Programs 2008, EPTCS 1, 2009, 123-129.
(available at `http://arxiv.org/abs/0906.3257v1`).

29. Lafitte G. and Papazian C. (2007)
The fabric of small Turing machines
in: *Computation and Logic in the Real World, Proceedings of the Third Conference on Computabiliy in Europe*, June 2007, 219–227 (available at `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.104.3021&rep=rep1&type=pdf#page=231`).

30. Lin S. (1963)
Computer studies of Turing machine problems
Ph.D. Thesis, The Ohio State University, Columbus, 1963.

31. Lin S. and Rado T. (1965)
Computer studies of Turing machine problems
Journal of the ACM **12** (2), April 1965, 196–212.

32. Ludewig J. , Schult U. and Wankmüller F. (1983)
Chasing the busy beaver. Notes and observations on a competition to find the 5-state busy beaver
Forschungsberichte des Fachbereiches Informatik Nr 159, Universität Dortmund, 1983 (63 pages)
(available at `http://elib.uni-stuttgart.de/bitstream/11682/8466/1/lud20.pdf`).

33. Lynn D.S. (1972)
New results for Rado's sigma function for binary Turing machines
IEEE Transactions on Computers C-**21** (8), August 1972, 894–896.

34. Machlin R. and Stout Q.F. (1990)
The complex behavior of simple machines
Physica D **42**, June 1990, 85–98.

35. Marxen H. and Buntrock J. (1990)
Attacking the Busy Beaver 5
Bulletin of the EATCS No 40, February 1990, 247–251
(available at `http://www.drb.insel.de/~heiner/BB/mabu90.html`).

36. Michel P. (1993)
Busy beaver competition and Collatz-like problems
Archive for Mathematical Logic **32** (5), 1993, 351–367.

37. Michel P. (2004)
Small Turing machines and generalized busy beaver competition
Theoretical Computer Science **326** (1-3), October 2004, 45–56.

38. Michel P. (2010)
Homology of groups and third busy beaver function
International Journal of Algebra and Computation **20** (6), 2010, 769–791.

39. Michel P. (2015)
Problems in number theory from busy beaver competition
Logical Methods in Computer Science **11** (4:10), 2015, 1–35
(available at `http://arxiv.org/pdf/1311.1029.pdf`).

40. Nabutovsky A. and Weinberger S. (2007)
Betti numbers of finitely presented groups and very rapidly growing functions
Topology **46** (2), March 2007, 211–223.

41. Oberschelp A., Schmidt-Göttsch K. and Todt G. (1988)
Castor quadruplorum
Archive for Mathematical Logic **27** (1), 1988, 35–44.

42. Petersen H. (2006)
Computable lower bounds for busy beaver Turing machines
Studies in Computational Intelligence **25**, 2006, 305–319.

43. Petersen H. (2017)
Busy beaver scores and alphabet size
in: *Proceedings of the 21st International Symposium on Fundamentals of Computation Theory, FCT 2017*, Lecture Notes in Computer Science Vol. 10472, Springer, 2017, 409–417
(available at `http://arxiv.org/abs/1704.08752`).

44. Rado T. (1962)
On non-computable functions
Bell System Technical Journal **41** (3), May 1962, 877–884.

45. Rado T. (1963)
On a simple source for non-computable functions
in: *Proceedings of the Symposium on Mathematical Theory of Automata, April 1962*, Polytechnic Institute of Brooklyn, April 1963, 75–81.

46. Soare R.I. (1996)
Computability and recursion
Bulletin of Symbolic Logic **2**, 1996, 284–321.

47. Soare R.I. (2007)
Computability and incomputability
in: *Proceedings of the 3rd CiE*, LNCS 4497, Springer, 2007, 705–715.

48. Soare R.I. (2009)
Turing oracle machines, online computing, and three displacements in computability theory
Annals of Pure and Applied Logic **160** (3), September 2009, 368–399.

49. Walsh T.R.S. (1982)
The busy beaver on a one-way infinite tape
SIGACT News **14** (1), Winter 1982, 38–43.

50. Wang K. and Xu S. (1995)
New relation between the shift function and the busy beaver function
Chinese Journal of Advanced Software Research **2** (2), 1995, 192–197.

51. Weimann B. (1973)
Untersuchungen über Rado's Sigma-Funktion und eingeschränkte Halteprobleme bei Turingmaschinen (German)
[Studies about Rado's Sigma function and limited halting problems of Turing machines]
Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, November 1973.

52. Weimann B., Casper K. and Fenzl W. (1973)
Untersuchungen über haltende Programme für Turing-Maschinen mit 2 Zeichen und

bis zu 5 Befehlen (German)
[Studies about halting programs for Turing machines with 2 symbols and up to 5 states] in: *Deussen P. (eds) GI. Gesellschaft für Informatik e. V. 2. Jahrestagung (Karlsruhe, October 1972)*, Lecture Notes In Economics and Mathematical Systems, Vol. 78, Springer, 1973, 72–81.

53. Yang R., Ding L. and Xu S. (1997)
Some better results estimating the shift function in terms of busy beaver function
SIGACT News **28** (1), March 1997, 43–48.

54. Yedidia A. and Aaronson S. (2016)
A relatively small Turing machine whose behavior is independent of set theory
Complex Systems **25** (4), 2016, 297–327
(available at `http://www.complex-systems.com/pdf/25-4-4.pdf`).

Websites

1. Scott Aaronson: `http://www.scottaaronson.com/writings/bignumbers.html`
(how to write big numbers with few symbols).

2. Joan Baez: `https://johncarlosbaez.wordpress.com/2016/05/21/the-busy-beaver-game`
(a blog).

3. Allen H. Brady: `http://www.cse.unr.edu/~al/BusyBeaver.html`
(Busy Beaver Competition for machines with 3 states and 3 symbols).

4. Georgi Gochev (Skelet): `http://skelet.ludost.net/bb`
(Busy Beaver Competition for machines with 5 states and 2 symbols, and for reversal machines).

5. Googology-wiki: `http://googology.wikia.com/wiki/Busy_beaver_function`
(a good introduction).

6. James Harland: `http://goanna.cs.rmit.edu.au/~jah/busybeaver`
(definitions of some inverse functions for the busy beaver functions).

7. Tim Hutton: `https://github.com/GollyGang/ruletablerepository/wiki/TwoDimensionalTuringMachines`
(two-dimensional and higher-dimensional Turing machines)

8. P. Machado and F. Pereira: `http://fmachado.dei.uc.pt/publications`
(Turing machines defined by 4-tuples).

9. Heiner Marxen: `http://www.drb.insel.de/~heiner/BB`
(the top web reference on busy beavers).

10. Pascal Michel: `http://www.logique.jussieu.fr/~michel`
(from which the present paper is written).

11. B. van Heuveln: http://www.cogsci.rpi.edu/~heuveb/Research/BB/index.html
(Turing machines defined by 4-tuples).

12. H. J. M. Wijers: http://www.win.tue.nl/~wijers/bb-index.htm
(a comprehensive bibliography).

13. Wikipedia: http://en.wikipedia.org/wiki/Busy_beaver
(a good introduction).

14. “Wythagoras”: http://googology.wikia.com/wiki/User:Wythagoras/Rado's_sigma_function
(many lower bounds of $\Sigma(n, m)$ for high values of n and m).

15. Hector Zenil: http://demonstrations.wolfram.com/BusyBeaver
http://mathrix.org/zenil/
(demonstration on the Wolfram Demonstrations Project, and website).